MA342A 2017–18

Chapter 3: LCA groups and their duals

Contents

3.1 LCA groups

In this section we give a definition of a class of locally compact abelian groups G for which

(a) the examples discussed in Chapter 1 of $G = \mathbb{T}$, $G = \mathbb{Z}$, $G = \mathbb{R}$ and G finite all satisfy the definition

(b) the Fourier theory works nicely in the form of the Fourier transform.

We want the dual group $\hat{G}$ to still belong in the class in a natural way. For this we need to define an appropriate metrizable topology on $\hat{G}$.

In the examples of groups G from Chapter 1, we defined in each case the Fourier transform of $F\colon G \to \mathbb{C}$ as a function $\hat{F}\colon \hat{G} \to \mathbb{C}$, where $\hat{F}$ is defined either as an integral or a sum. (In the case of $G = \mathbb{Z}$, we did not actually give the definition.) In the case $G = \mathbb{T}$ and $\hat{G} = \mathbb{Z}$ we discussed in Chapter 2 results concerning recovering F from $\hat{F}$ (as the Fourier series, which is a sum over $\hat{G} = \mathbb{Z}$).

Sums may be viewed as integrals with respect to counting measure and in the general case we need a good measure on G in order to define the Fourier transform. We need a similarly good measure on $\hat{G}$ in order to come up with a version of the Fourier series. It will be replaced in general by an integral over $\chi \in \hat{G}$ of $\hat{F}(\chi)\chi$. The concept of a "good measure" we want is called Haar measure on the group.

We repeat some definitions from Chapter 1 for ease of reference.

3.1.1 Definition (Definition 1.3.3)**.** A metric space (X, d) is called *locally compact* if for each point $x_0 \in X$ there is some $r > 0$ (depending on x_0) such that the closed ball $\bar{B}(x_0, r)$ is compact.

We also use the term locally compact for metrizable spaces.

3.1.2 Definition (Definition 1.3.5)**.** By a *topological group* we will mean a group G which is also a metrizable space where multiplication and inversion are continuous.

In more detail, we suppose d is a metric on G that gives rise to the topology, and then we defined a metric on $G \times G$ by $\rho((g_1, g_2), (h_1, h_2)) = d(g_1, h_1) + d(g_2, h_2)$. Then $G \times G$ is a metrizable space. We insist then that the maps

(TG1) multiplication : $G \times G \to G$ ($(g, h) \mapsto gh$)

(TG2) inversion : $G \to G$ ($g \mapsto g^{-1}$)

are each continuous.

3.1.3 Definition (Base of a topological space)**.** If X is a topological space (or a metrizable space), then a subfamily $\mathcal{B}$ of the open sets is called a *base (for the open sets)* of the topology if

$$x \in U \subseteq X, U \text{ open} \Rightarrow \exists B \in \mathcal{B} \text{ with } x \in B \subset U$$

3.1.4 Definition (second countable space)**.** A topological space (or metrizable space) X is called *second countable* if there exists a base $\mathcal{B}$ for the topology where $\mathcal{B}$ is a countable collection of sets.

3.1.5 Example. $X = \mathbb{R}^N$ is second countable (any $N \in \mathbb{N}$)

Proof. Consider $\mathcal{B} = \{B(q, 1/n) : q \in \mathbb{Q}^N, n \in \mathbb{N}\}$. This is countable, consists of open sets (open balls in fact) and we now show it is a base.

We take balls in the usual Euclidean metric d.

If $x \in U \subseteq \mathbb{R}^N$ with U open, then there is $r > 0$ with $B(x, r) \subseteq U$. Choose $n \in \mathbb{N}$ with $1/n < r/2$. Choose $q \in \mathbb{Q}^N$ with $d(x, q) < 1/n$. Then $x \in B(q, 1/n) \subset B(x, 2/n) \subset B(x, r) \subseteq U$. □

If $Y \subset \mathbb{R}^N$ is any subset, then Y is also second countable (with the usual topology arising from (the restriction to Y of) d).

Proof. With $\mathcal{B}$ a countable base for $\mathbb{R}^N$ (for instance as above), $\{B \cap Y : B \in \mathcal{B}\}$ is a counatble base for Y. □

In particular $\mathbb{Z}$ and $\mathbb{T}$ are second countable.

3.1.6 Lemma. *If X is a locally compact and second countable metrizable space, then there is a countable base $\mathcal{B}$ for the topology such that each set $B \in \mathcal{B}$ has compact closure.*

Proof. Let $\mathcal{B}_0$ be a countable base for X. Let d be a metric on X that gives the topology. For each $x \in X$, there is a radius $r_x > 0$ such that the closed ball $\bar{B}(x, r_x)$ is compact. Since x is a point in the open ball $B(x, r_x)$, there is $B_x \in \mathcal{B}_0$ with $x \in B_x \subseteq B(x, r_x)$. Such a B_x has closure in the closed ball $\bar{B}(x, r_x)$ and so is compact.

In fact, if $0 < r < r_x$, using $x \in B(x, r)$ open, there is $B_{x,r} \in \mathcal{B}_0$ with $x \in B_{x,r} \subseteq B(x, r)$ and such a $B_{x,r}$ also has compact closure.

If we take $\mathcal{B} = \{B \in \mathcal{B}_0 : \bar{B} \text{ is compact}\}$, then $\mathcal{B}$ is countable and is a base for X because if we have $x \in U$ open, then there is $r \leq r_x$ with $B(x, r) \subseteq U$ and so $x \in B_{x,r} \subseteq U$ (and $B_{x,r} \in \mathcal{B}$). □

3.1.7 Lemma. *If X is a second countable locally compact metrizable space then there is a sequence $(K_n)_{n=1}^\infty$ of compact subsets of X with the following properties:*

(Ex1) $K_n \subseteq K_{n+1}^\circ$ *for* $n = 1, 2, \ldots$*;*

(Ex2) If $K \subseteq X$ *is compact , then there is* n *with* $K \subseteq K_n$*;*

(Ex3) $\bigcup_{n=1}^\infty K_n = X$.

(We refer to such a sequence $(K_n)_{n=1}^\infty$ *as an* exhaustion of X by compact sets.*)*

Proof. By Lemma 3.1.6, we know that X has a countable base $\mathcal{B} = \{B_1, B_2, \ldots\}$ such that each $\overline{B_n}$ is compact.

Then $\bigcup_{n=1}^\infty B_n = X$ because if $x \in X$, then there must be a basic open set B_n with $x \in B_n \subseteq X$.

Let $K_1 = \overline{B_1}$.

Since K_1 is compact and contained in $\bigcup_{n=1}^\infty B_n$ (that is the sets B_n form an open cover of K_1) there is some finite n_1 with $K_1 \subseteq \bigcup_{n=1}^{n_1} B_n$. We suppose that $n_1 \geq 2$. Take $K_2 = \bigcup_{n=1}^{n_1} \overline{B_n}$. Then K_2 is compact since it is a finite union of compact sets. We have

$$K_1 \subseteq \bigcup_{n=1}^{n_1} B_n \subseteq K_2$$

and that finite union is open. So $K_1 \subseteq K_2^\circ$.

We proceed inductively. That is, if we already have K_r, find $n_r \geq r+1$ such that

$$K_r \subseteq \bigcup_{n=1}^{n_r} B_n$$

Put

$$K_{r+1} = \bigcup_{n=1}^{n_r} \overline{B_n}$$

If $K \subseteq X$ is compact, then there is r with $K \subseteq \bigcup_{n=1}^{r} B_n$. Then $K \subseteq K_r$ since $n_r > r$.

Also $X = \bigcup_{n=1}^\infty B_n \subseteq \bigcup_{r=1}^\infty K_r$ because $B_n \subseteq K_n$.

This shows that the sequence $(K_r)_{r=1}^\infty$ is an exhaustion. $\square$

3.1.8 Example. The groups G we saw in Chapter 1 are all second countable and locally compact metrizable spaces and so have an exhaustion by the above. But in each case, we can easily write one down:

$G = \mathbb{T}$: put $K_n = \mathbb{T}$ for all n

$G = \mathbb{R}$: put $K_n = [-n, n]$

$G = \mathbb{Z}$: put $K_n = \{-n, -(n-1), \ldots, n-1, n\}$

G finite abelian: put $K_n = G$ for all n

3.1.9 Definition. An *LCA group* is an abelian topological group that is locally compact and second countable.

We recall now from Chapter 1:

3.1.10 Definition (Characters, Definition 1.3.7)**.** If G is a an abelian topological group, then a *character* of G is a continuous group homomorphism $\chi\colon G \to \mathbb{T}$.

3.1.11 Definition (Dual group)**.** If G is an abelian topological group, we define $\hat{G}$ to be the set of all characters $\chi\colon G \to \mathbb{T}$ and introduce a multiplication rule for characters χ_1, χ_1 by

$$(\chi_1\chi_2)(g) = \chi_1(g)\chi_2(g)$$

3.1.12 Notation. Let G be an LCA group and let $(K_n)_{n=1}^\infty$ be a fixed exhaustion of X by compact sets.

We define $d_n\colon \hat{G} \times \hat{G} \to [0, \infty)$ by

$$d_n(\chi_1, \chi_2) = \sup_{g \in K_n} |\chi_1(g) - \chi_2(g)|$$

(Note that $0 \leq d_n(\chi_1, \chi_2) \leq 2$ because $\chi_1(g), \chi_2(g) \in \mathbb{T}$.)

3.1.13 Lemma. *We can define a metric $d\colon \hat{G} \times \hat{G} \to [0, \infty)$ by*

$$d(\chi_1, \chi_2) = \sum_{n=1}^{\infty} \frac{1}{2^n} d_n(\chi_1, \chi_2)$$

Proof. We will not give this proof, but is is quite straightforward. The functions d_n may not be metrics because $d_n(\chi_1, \chi_2) = 0$ may not imply $\chi_1 = \chi_2$, but d_n obeys the other properties for a metric. □

3.1.14 Lemma. *If $(\chi_n)_{n=1}^\infty$ is a sequence in $\hat{G} =$ and $\chi \in \hat{G}$, then*

$$\lim_{n \to \infty} \chi_n = \chi \text{ in the metric space } (G, d)$$

if and only if

$$\lim_{n \to \infty} \sup_{g \in K} |\chi_n(g) - \chi(g)| = 0$$

holds for each compact $K \subseteq G$.

(That is convergence of sequences in the metric d is equivalent to uniform convergence on (all) compact subsets of G.)

Proof. Again, this is not very complicated but we will not give the details. □

3.1.15 Corollary. *The topology on $\hat{G}$ given by the metric d does not depend on the choice of the fixed exhaustion of X by compact sets, that is it does not depend on $(K_n)_{n=1}^\infty$ (though the metric d does depend on that choice).*

Proof. This is immediate from Lemma 3.1.14 □

Recall (Proposition 1.3.10) that $\hat{G}$ is an abelian group (if G is a topological abelian group).

3.1.16 Theorem. *If G is an LCA group, then $\hat{G}$ with the topology arising from the metric d is also an LCA group.*

Proof. This requires quite a bit of proof. We refer to chapter 6 of A. Deitmar, *A First Course in Harmonic Analysis*, Springer Universitext (2002), but skip it. □

3.1.17 Theorem. *(i) $\hat{\mathbb{R}}$ can be identified with $\mathbb{R}$*

(ii) $\hat{\mathbb{Z}}$ can be identified with $\mathbb{T}$

(iii) $\hat{\mathbb{T}}$ can be identified with $\mathbb{Z}$

(iv) If G is a finite abelian group (with the discrete topology), then also $\hat{G}$ is a finite abelian group of the same order (also with the discrete topology).

Proof. The point here is that, in chapter 1, where we showed already that if G is one of these groups, then $\hat{G}$ is as claimed as a group. We now assert the metric d gives the usual topology on $\hat{G}$. Again we will not prove it here. □

3.1.18 Theorem (Pontryagin duality)**.** *If G is an LCA group, then every character of $\hat{G}$ is of the form $\hat{g}\colon \hat{G} \to \mathbb{T}$ given by*

$$\hat{g}(\chi) = \chi(g)$$

for some $g \in G$.

The map $g \mapsto \hat{g}$ from G to $\hat{\hat{G}}$ is a homeomorphism.

(This is usually summarized as $\hat{\hat{G}} = G$.)

Again we do not prove this.

3.2 Haar measure

3.2.1 Definition. If G is an LCA group, then the *Borel σ-algebra* on G is the smallest σ-algebra Σ_{Borel} of subsets of G such that every open subset of G is in Σ_{Borel}.

3.2.2 Definition. If G is an LCA group, then a *Borel measure* on G is a measure $\lambda\colon \Sigma_{\text{Borel}} \to [0, \infty]$.

(That means $\lambda(\emptyset) = 0$ and λ is countably additive — see Definition 2.A.1.3.)

3.2.3 Definition. A Borel measure λ on an LCA group G is called *inner regular* if $\lambda(K) < \infty$ whenever $K \subseteq G$ is compact and if

$$\lambda(U) = \sup\{\lambda(K) : K \subseteq U \text{ and } K \text{ compact}\}$$

holds whenever $U \subseteq G$ is open.

3.2.4 Definition. A Borel measure λ on an LCA group G is called *outer regular* if

$$\lambda(E) = \inf\{\lambda(U) : U \text{ open and } E \subseteq U\}$$

holds for all $E \in \Sigma_{\text{Borel}}$.

3.2.5 Definition. A Borel measure λ on an LCA group G is called *translation invariant* if it satisfies

$$\lambda(sE) = \lambda(E) \text{ for all } E \in \Sigma_{\text{Borel}}, s \in G$$

where $sE = \{sg : g \in E\}$. (Note: if the group operation is written $+$ we should write $s + E = \{s + g : g \in E\}$ instead.)

3.2.6 Definition. A *Haar measure* on an LCA group G is nonzero Borel measure λ on G which is both inner and outer regular and also translation invariant.

3.2.7 Theorem. *Every LCA group G has a Haar measure λ_G and every other Haar measure on G is a positive multiple of λ_G. (In other words, λ_G is unique apart from rescaling.)*

The proof is quite difficult. (Well, there is more than one proof, but none are very simple.)

3.2.8 Examples. (i) If $G = \mathbb{R}$, Lebesgue length measure is a Haar measure on G.

(ii) If $G = \mathbb{T}$, Lebesgue arc length measure is a Haar measure on G. We have been using $1/(2\pi)$ times that, that is normalizing $\lambda_{\mathbb{T}}$ so that $\lambda_{\mathbb{T}}(\mathbb{T}) = 1$.

Thus we take $\lambda_{\mathbb{T}}$ to be a probability measure.

If G is any compact LCA group, we can assume that $\lambda_G(G) = 1$ (and this is a common normalization to use).

(iii) If $G = \mathbb{Z}$, Haar measure $\lambda_{\mathbb{Z}}$ must give a positive weight to the one point set $\{0\}$ because translation invariance forces

$$\lambda_{\mathbb{Z}}(\{n\}) = \lambda_{\mathbb{Z}}(\{0\} + n) = \lambda_{\mathbb{Z}}(\{0\})$$

and if $\lambda_{\mathbb{Z}}(\{0\} = 0$, then we would have

$$\lambda_{\mathbb{Z}}(\mathbb{Z}) = \lambda_{\mathbb{Z}}\left(\bigcup_{n\in\mathbb{Z}}(\{n\}\right) = \sum_{n\in\mathbb{Z}} \lambda_{\mathbb{Z}}(\{n\}) = 0$$

We typically take $\lambda_{\mathbb{Z}}(\{0\}) = 1$ and then $\lambda_{\mathbb{Z}}$ is counting measure, $\lambda_G(E)$ is the number of elements in E for all $E \subseteq \mathbb{Z}$.

This is a normalization that is commonly used when G has the discrete topology.

(iv) If G is a finite abelian group with N elements, then G is both compact and discrete. So according to the last two examples there are two "obvious" normalizations to choose. One is the make $\lambda_G(G)$ be 1, and that would mean taking λ_G to be $1/N$ times counting measure. The other would be to take counting measure.

In Chapter 1, we defined the Forurier transform using what amounts to $1/\sqrt{N}$ times counting measure. We did the same on $\hat{G}$ and we got a nice formula for functions $f : G \to \mathbb{C}$ in terms of $\hat{f}$ (Corollary 1.4.13).

3.3 Fourier transform

3.3.1 Definition. If G is an LCA group and λ_G is a Haar measure on G, then we define the *Fourier transform* of a function $F\colon G \to \mathbb{C}$ to be the function $\hat{F}\colon \hat{G} \to \mathbb{C}$ given by

$$\hat{F}(\chi) = \int_G F\bar{\chi}\,d\lambda_G = \int_{g\in G} F(g)\overline{\chi(g)}\,d\lambda_G(g)$$

(However we need to assume that the integral makes sense first. We can most naturally assume that F is integrable with respect to the measure λ_G, that is that F is Borel measurable and that $\int_G |F|\,d\lambda_G < \infty$. We write $F \in L^1(G)$ for this, though we should really say $F \in \mathcal{L}^1(G)$ and then define $L^1(G)$ as the space of λ_G-almost everywhere equivalence classes of functions in $\mathcal{L}^1(G)$.)

3.3.2 Lemma. *If $F\colon G \to \mathbb{C}$ is continuous and $\{g \in G : F(g) \neq 0\}$ has compact closure K in G, then $F \in L^1(G)$.*

(We write $C_c(G)$ for the F satisfying these conditions.)

Proof. Since F is continuous it must be Borel measurable.

Since K is compact and $|F|$ is continuous on K, it has a largest value M (say) on K. So $|F(g)| \leq M\chi_K(g)$ holds for all $g \in G$ (where we now use χ_K for the characteristic function of K). Since λ_G is inner regular, $\lambda_G(K) < \infty$ and

$$\int_G M\chi_K(g)\,d\lambda_G(g) = M\lambda_G(K) < \infty$$

So

$$\int_G |F(g)|\,d\lambda_G(g) \leq \int_G M\chi_K(g)\,d\lambda_G(g) < \infty$$

and $F \in L^1(G)$. □

3.3.3 Definition. $\mathcal{L}^2(G)$ is the space of Borel measurable $F\colon G \to \mathbb{C}$ which are square-integrable, that is satisfy

$$\int_G |F|^2\,d\lambda_G = \int_G |F(g)|^2\,d\lambda_G(g) < \infty$$

$L^2(G)$ is the space of λ_G-almost everywhere equivalence classes of functions $F \in \mathcal{L}^2(G)$ (where we say F and H are equivalent if $\lambda_G(\{g \in G : F(g) \neq H(g)\}) = 0$).

3.3.4 Proposition. *We can define a norm on $L^2(G)$ by*

$$\|F\|_2 = \left(\int_G |F|^2\,d\lambda_G\right)^{1/2}$$

and then $L^2(G)$ is a Banach space.

Moreover $C_c(G) \subset L^2(G)$ is a dense subspace.

3.3.5 Theorem (Plancherel theorem)**.** *Let G be an LCA group and λ_G a chosen Haar measure on G. Then is a choice of a Haar measure $\lambda_{\hat{G}}$ on $\hat{G}$ such that $\|\hat{F}\|_2 = \|F\|_G$ holds for all $F \in C_c(G)$.*

With these choices the Fourier transform $F \mapsto \hat{F}$ extends to a surjective linear isometry from $L^2(G)$ to $L^2(\hat{G})$.

Every $F \in L^2(G)$ can be recovered from $\hat{F} \in L^2(\hat{G})$. The inverse Fourier transform of $H \in L^2(\hat{G})$ is defined for $H \in C_c(\hat{G})$ by

$$\check{H}(g) = \int_{\chi \in \hat{G}} H(\chi)\chi(g)\, d\lambda_{\hat{G}}(\chi)$$

and this also extends to all $H \in L^2(\hat{G})$ in such a way that $H \mapsto \check{H}$ is a surjective linear isometry $L^2(\hat{G}) \to L^2(G)$ with $\|\check{H}\|_2 = \|H\|_2$.

The fact that it is the inverse means $\check{\hat{F}} = F$ for $F \in L^2(G)$.

The proof of this is not so easy, even for the case $G = \mathbb{R}$ and $\hat{G} = \mathbb{R}$. One bothersome issue in the case of $\mathbb{R}$ (or groups where $\lambda_G(G) = \infty$) is that $L^2(G)$ is not contained in $L^1(G)$. So the integral we used to define the Fourier transform does not make sense for general $F \in L^2(G)$ and we have to work to come up with the right choice for $\hat{F}$.

In the case of compact groups like $G = \mathbb{T}$, we do have $L^2(G) \subset L^1(G)$ and that makes life easier. But if $G = \mathbb{T}$, then $\hat{G} = \mathbb{Z}$ is not compact and we have this issue for the inverse transform.

Unfortunately, we do not have time to deal with these issues and many results in this chapter have not been proved at all. We also did not discuss applications of Fourier theory (such as in PDEs, sound waves, image compression) or algorithms such as the Fast Fourier Transform (for efficent computation of the Fourier transform on finite abelian groups, especially cyclic groups of order 2^n).

Nov 30: Fix typo in statement of Theorem 3.1.18. Apr 30: Fix typos in Definitions 3.2.3 and 3.2.4

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