HOLY GRAIL workshop

Homology, graded algebras, infinity and Lie type structures

12-13 October, 2020

Dedicated to the memory of Victor Nikolaevich Latyshev

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Schedule

!!! Note: the time you see here is LONDON TIME!!!

It means in central Europe it will be +1Hour, in Moscow +2Hour, in Novosibirsk +6Hours, in Boston -5Hours, in Sao Paulo -4Hours, in San Diego -8Hours!

Monday, 12 October

t+6 **10.00-10.10** Leonid Bokut *Few opening words about V.N. Latyshev and the Shirshov school*

$t+1$ **10.10-11.00** Maxim Kontsevich *Noncommutative Poisson brackets revisited*

$t+0$ **11.10-12.00** Michael Wemyss *Contraction algebras, plumbings and flops*

$t+0$ **12.10-13.00** Andrey Lazarev *Categorical Koszul duality*

13.10-14.10 *Lunch*

$t+1$ **14.10-15.00** Borya Shoikhet *The twisted tensor product of dg categories and a contractible 2-operad*

$t+0$ **15.10-16.00** Travis Schedler *Multiplicative preprojective algebras are 2-Calabi–Yau*

16.00-16.20 *Coffee Break*

$t+1$ **16.20-16.50** David Fernández *Noncommutative Poisson geometry and pre-Calabi-Yau algebras*

$t+1$ **16.50-17.20** Estanislao Herscovich *Double quasi-Poisson algebras and pre-Calabi-Yau algebras*

$t+0$ **17.30-18.20** Sue Sierra *The Poisson spectrum of the symmetric algebra of the Virasoro algebra*

Tuesday, 13 October

$t+1$ **10.00-10.50** Oksana Yakimova *Symmetrisation and the Feigin–Frenkel centre*

$t-3$ **10.50-11.20** Victor Petrogradsky *Nil restricted Lie algebras of oscillating intermediate growth*

$t+2$ **11.20-11.30** Aleksey Kanel-Belov *About Victor Nikolaevich Latyshev and his school*

$t+2$ **11.30-12.00** Igor Melnikov *On cogrowth function of uniformly recurrent sequences*

$t+2$ **12.00-12.50** Tanya Gateva *Associative algebras and Lie algebras defined by Lyndon words*

$t-4$ **12.50-13.00** Ivan Shestakov *Memories about Victor Nikolaevich Latyshev*

13.00-14.00 *Lunch*

$t-5$ **14.00-14.50** Vladimir Retakh *Noncommutative Catalan numbers and Hankel matrices*

$t-5$ **15.00-15.50** Arkady Berenshtein *Noncommutative orthogonal polynomials and beyond*

15.50-16.10 *Coffee Break*

$t+2$ **16.10-16.40** Ilya Ivanov-Pogodaev *The construction of infinite finitely presented nilsemigroup*

$t+2$ **16.40-17.10** Pavel Kolesnikov *On the Gröbner–Shirshov bases of conformal and vertex algebras*

$t+2$ **17.10-17.40** Stanislav Shkarin *On possible Hilbert series*

$t+2$ **17.40-18.10** Vsevolod Gubarev *Semisimple post-Lie algebra structures*

$t-8$ **18.10-19.00** Efim Zelmanov *Growth functions*

HOLY GRAIL workshop
Abstracts

Arkady Berenshtein

Noncommutative orthogonal polynomials and beyond

The goal of my talk, based on joint work (still in progress) with Vladimir Retakh, is to present new developments in the theory of orthogonal polynomials over any rings originated in the work of Vladimir with I.M. Gelfand and collaborators.

Our approach can be summarized as follows. A (generalized orthogonal) basis $\pi = \pi_0(t) = 1$, $\pi_1(t), \pi_2(t), \ldots$ of a polynomial ring $R[t]$ is a single eigenvector of a certain semi-infinite row-finite matrix P over a ring R, with the eigenvalue t, i.e.,$P\pi = \pi t$. We argue that for such π to exist it is necessary to find a (semi-infinite row-finite) matrix M such that $PM = ME$, where E is the semi-infinite Jordan block. If P has an additional property $P_{i,i+1} = 1$ and $P_{ij} = 0$ for $j > i + 1$, we prove that all such M are of the from $M_P T$, where T is a Toeplitz matrix and M_P is a unique lower unitriangular matrix whose inverse is determined by that its i-th row is the top row of P^i.

Of course, this P is a generalization of the Jacobi matrix of π and M_P determines π because the i-th coefficient of $\pi_j(t)$ is the (ji)th matrix entry of M_P. If P is, indeed, tridiagonal and D is a diagonal one such that PD is symmetric, then we recover the "classical" theory (both commutative and noncommutative) of orthogonal polynomials because the matrix $H_P := M_P^{-1} D M_P^{-T}$ is Hankel, moreover, its (ij)th entry is the left upper corner entry of $P^{i+j}D$. The matrix H_P is, indeed, the Gramm matrix of the inner product on $R[t]$.

Quite surprisingly, for a certain tridiagonal matrix P over the free ring R in infinitely many generators $y_1, y_2, ..$, the Hankel matrix H_P consists of noncommutative analogues of Catalan numbers, which Vladimir and I introduced in 2017. This example is rather universal: specializing $y_1, y_2, \ldots$ to any elements of any field K, this particular P becomes an arbitrary Jacobi matrix over K with $J_{i,i+1} = 1$, thus π becomes any monic orthogonal basis in $K[t]$.

David Fernández

Noncommutative Poisson geometry and pre-Calabi-Yau algebras

This is the first part of a mini-series of 2 talks jointly with Estanislao Herscovich.

Since the definition of (compact) Calabi-Yau structures on compact A_∞-algebras is too restrictive in applications related to Fano manifolds, open Calabi-Yau manifolds, Fukaya categories or path spaces, M. Kontsevich and Y. Vlassopoulos introduced pre-Calabi-Yau algebras, which can be regarded as Poisson structures in formal geometry.

Independently, M. Van den Bergh introduced double Poisson algebras to solve the long-standing problem of defining appropriate Poisson structures in the setting of noncommutative geometry. Quite strikingly, N. Iyudu and M. Kontsevich constructed an explicit bijection between pre-Calabi-Yau algebras and double Poisson algebras.

In this talk, we shall explain this bijection and its extension to the differential graded setting. Moreover, we will describe a further generalisation of these results to include double P_∞-algebras, as introduced by T. Schedler.

Tatiana Gateva-Ivanova

Associative algebras and Lie algebras defined by Lyndon words

Assume that $X = \{x_1, \cdots, x_g\}$ is a finite alphabet and $\mathbf{k}$ is a field. We study the class $\mathfrak{C}(X, W)$ of associative graded $\mathbf{k}$-algebras A generated by X and with a fixed obstructions set W consisting of Lyndon words in the alphabet X. Important examples are the monomial algebras $A = \mathbf{k}\langle X\rangle/(W)$, where W is an antichain of Lyndon words of arbitrary cardinality and the enveloping algebra $U\mathfrak{g}$ of any X-generated Lie $\mathbf{k}$-algebra $\mathfrak{g} = Lie(X)/([W])$, whenever the set of standard bracketings $[W] = \{[w] \mid w \in W\}$ is a Gröbner-Shirshov Lie basis. We prove that all algebras A in $\mathfrak{C}(X, W)$ share the same Poincaré-Birkhoff-Witt type $\mathbf{k}$-basis built out of the so called *Lyndon atoms* N (determined uniquely by W) but, in general, N may be infinite. Moreover, A has polynomial growth if and only if the set of Lyndon atoms N is finite. In this case A has a $\mathbf{k}$-basis $\mathfrak{N} = \{l_1^{\alpha_1} l_2^{\alpha_2} \cdots l_d^{\alpha_d} \mid \alpha_i \geqslant 0, 1 \leqslant i \leqslant d\}$, where $N = \{l_1, \cdots, l_d\}$. Surprisingly, in the case when A has polynomial growth its global dimension does not depend on the shape of its defining relations but only on the set of obstructions W. We prove that if A has polynomial growth of degree d then A has global dimension d and is standard finitely presented, with $d-1 \leqslant |W| \leqslant d(d-1)/2$. We study when the set of standard bracketings $[W] = \{[w] \mid w \in W\}$ is a Gröbner-Shirshov Lie basis. We use our general results to classify the Artin-Schelter regular algebras A generated by two elements, with defining relations $[W]$ and global dimension $\leqslant 7$.

References

[1] Tatiana Gateva-Ivanova, *Algebras defined by Lyndon words and Artin-Schelter regularity*, arXiv preprint arXiv:1905.11281 (2019).

[2] Tatiana Gateva-Ivanova, Gunnar Fløystad, *Monomial algebras defined by Lyndon words*, Journal of Algebra **403** (2014), 470–496.

[3] Tatiana Gateva-Ivanova, *Quadratic algebras, Yang–Baxter equation, and Artin–Schelter regularity*, Advances in Mathematics **230** (2012), 2152–2175.

[4] Tatiana Gateva-Ivanova, *Global dimension of associative algebras*, Applied Algebra, Algebraic Algorithms and Error-Correcting Codes, Lecture Notes in Computer Science, **357** (1989), 213–229.

Vsevolod Gubarev

Semisimple post-Lie algebra structures

Abstract. Let $(n, \{,\})$ be a semisimple finite-dimensional Lie algebra over the field of complex numbers. It is known that given a Rota-Baxter operator R of weight 1 on n, we get a new Lie algebra structure $(g, [,])$ on the same vector space under the bracket $[x, y] = \{R(x), y\} + \{x, R(y)\} + \{x, y\}$. We prove that if g is semisimple, then n and g are isomorphic.

Estanislao Herscovich

Double quasi-Poisson algebras and pre-Calabi-Yau algebras

This is the second part of a mini-series of 2 talks jointly with David Fernndez.

As it is the case for double Poisson algebras, double quasi-Poisson algebras were introduced by M. Van den Bergh as noncommutative analogues of quasi-Poisson manifolds, as defined by A. Alekseev, Y. Kosmann-Schwarzbach and E. Meinrenken. On the other hand, as exposed in the first talk of this mini-series, N. Iyudu and M. Kontsevich found a remarkable link between double Poisson algebras and pre-Calabi-Yau algebras, a notion introduced by Kontsevich and Y. Vlassopoulos.

The aim of this talk will be to explain how double quasi-Poisson algebras give rise to pre-Calabi-Yau algebras, in the same spirit as the connection found by Iyudu and Kontsevich. Nevertheless, this pre-Calabi-Yau structure is more intricate since it involves an infinite number of nonvanishing higher multiplications for the associated pre-Calabi-Yau algebra, which include the Bernoulli numbers.

Ilya Ivanov-Pogodaev

(joint with Alexey Kanel-Belov)
The construction of infinite finitely presented nilsemigroup

Burnside-type problems have made an enormous contribution to the development of modern algebra. This problem covered a wide range of issues, both in group theory and in related fields, stimulated algebraic research. The question of the local finiteness of groups with the identity $x^n = 1$ was answered negatively in the famous papers of Novikov and Adyan. After that, the value of the exponent was improved. The papers of Novikov and Adian had a tremendous influence on the work of Rips, who later developed the method of canonical form and constructed examples of infinite periodic groups with additional properties.

All available examples of infinite periodic groups are not finitely presented.

In particular, the question of the existence of a finitely presented infinite periodic group is open for both bounded and unbounded cases.

In rings and semigroups, Burnside-type questions concern objects with the nil property (for each element x there exists n such that $x^n = 0$).

The question posed by Latyshev about the existence of a finitely defined infinite nil-ring is also open. Few examples of finitely defined algebraic objects with Burnside properties are known.

So, it is interesting to discuss a method allows us to obtain such constructions.

The talk is devoted to the construction of a finitely defined infinite nilsemigroup satisfying the identity $x^9 = 0$. This construction answers the problem posed by Shevrin and Sapir.

Construction consists of three stages. At the first stage, a sequence of geometric complexes is determined. Every complex consist of several 4-cycles (squares) glued together.

At the second stage, the vertices and edges of each complex are encoded with the letters of the finite alphabet. The words of the semigroup will correspond to the encoding of paths on complexes. The local equivalences of two paths of length 2 connecting the opposite sides of each square of the complex correspond to the defining relations. In addition, monomial relations are introduced that lead to zero encoding of impossible paths, as well as paths back and forth along some edge of the complex.

At the third stage we introduce an algorithm that brings an arbitrary word to its canonical form. Each word locally represents a path encoding on a complex. In this case, the application of the defining relations leads to a local transformation of this path, which does not change its ends and its length. In the process of local transformations, a forbidden subword may appear in the word, in this case it is reduced to zero. Otherwise, we end up with an embedding of this path in some complex.

The complexes need the following properties:

1. Uniform ellipticity. Any path can be changed quite strongly by local transformations;

2. Local finiteness. The whole family of complexes admits coloring of vertices and edges in a finite set of colors (letters)

3. Determinism. If the colors of the edges and vertices along the path of length 2 connecting the opposite vertices of a certain square, are known, then the colors of the edges and vertices along the path paired to it (along the other two sides) are also calculated. This property allows us to correctly introduce the defining relations.

4. Aperiodicity. The path encodings on the complex do not contain periodic words of period 9.

As a result, nonzero words in the semigroup correspond to the shortest paths on the complexes. The encodings corresponding to not shortest and impossible paths are reduced to zero, leaving an infinite set of nonequivalent non-zero paths.

Pavel Kolesnikov

On the Gröbner–Shirshov bases of conformal and vertex algebras

Various algebraic structures may be defined by generators and relations. Finding normal forms, solving the word problem in such systems is a common problem appearing in algebraic studies. There is a routine approach based on the notion of a Gröbner–Shirshov basis which often helps to investigate a structure of a system defined by generators and relations. We will consider a modification of this approach which allows us to reduce the combinatorial problems for vertex algebras and associative conformal algebras to the "ordinary" systems.

Maxim Kontsevich

Noncommutative Poisson brackets revisited

Andrey Lazarev

Categorical Koszul duality

(joint with J. Holstein)

Differential graded (dg) Koszul duality is a certain adjunction between the category of dg algebras and conilpotent dg coalgebras that becomes an equivalence on the levels of homotopy categories. More precisely, this adjunction is a Quillen equivalence of the corresponding closed model categories. Various versions of this result exist and play important roles in rational homotopy theory, deformation theory, representation theory and other related fields. We extend it to a Quillen equivalence between dg categories (generalizing dg algebras) and a class of dg coalgebras, more general than conilpotent ones. As applications we describe explicitly and conceptually Luries dg nerve functor as well as its adjoint and characterize derived categories of $(\infty, 1)$-categories as derived categories of comodules over simplicial chain coalgebras.

Igor Melnikov, Ivan Mitrofanov

(sci. advisor Alexey Kanel-Belov)

On cogrowth function of uniformly recurrent sequences

An innite word A is called uniformly recurrent, if for any subword S of the word A there exists n, such that for any subword W of the word A of length n, S is subword of W. An obstruction is a word which is not a subword of A, but whose any subword is also a subword of A. Let $OW(n)$ be the number of obstructions with length no more than n. We will show that $\overline{\lim\limits_{n\to\infty}} \frac{OW(n)}{\log_3 n} \geqslant 1$.

Igor Melnikov, and Ivan Mitrofanov On cogrowth function of uniformly recurrent sequences. https://arxiv.org/abs/2001.02272

Victor Petrogradsky

Nil restricted Lie algebras of oscillating intermediate growth

Different versions of Burnside Problem ask what one can say about finitely generated periodic groups under additional assumptions. For associative algebras, Kurosh type problems ask similar questions about properties of finitely generated nil (more generally, algebraic) algebras. Similarly, one considers finitely generated restricted Lie algebras with a nil p-mapping.

Recently, the question how algebras grow became popular [1]. Groups of oscillating growth were constructed in [2]. Now we study oscillating intermediate growth in the class of nil restricted Lie algebras.

Theorem [3,4] For any prime p we construct a family of 3-generated restricted Lie algebras of intermediate oscillating growth, called Phoenix algebras, with the properties:

1. For infinitely many integers n, the algebra is almost dying by having a quasi-linear growth of type $n\big(\underbrace{\ln\ldots\ln}_{q} n\big)^{\kappa}$, where $q \in \mathbb{N}$, $\kappa \in \mathbb{R}^{+}$ are arbitrary fixed constants.

2. For infinitely many integers n the algebra is resuscitating by having intermediate growth of type $\exp(n/(\ln n)^{\lambda})$, where $\lambda = \lambda(p)$ is a constant.

3. The growth stays always between two functions of the type above.

4. These restricted Lie algebras have a nil p-mapping.

References

1. Bell J., Zelmanov E., On the growth of algebras, semigroups, and hereditary languages. arXiv:1907.01777.

2. Kassabov M., Pak, I., Groups of oscillating intermediate growth. *Ann. Math.* (2) **177**, (2013) No. 3, 1113–1145.

3. Petrogradsky V., Clover nil restricted Lie algebras of quasi-linear growth, arXiv:2004.01713.

4. Petrogradsky V., Nil restricted Lie algebras of oscillating intermediate growth, arXiv:2004.05157.

Vladimir Retakh

Noncommutative Catalan numbers and Hankel matrices

This is an introduction to the joint work with Arkady Berenstein presented at this conference. Our goal is to introduce and study noncommutative Catalan numbers C_n which belong to the free Laurent polynomial algebra in n generators. Our noncommutative numbers admit interesting (commutative and noncommutative) specializations, one of them related to Garsia-Haiman (q, t)-versions, another – to solving noncommutative quadratic equations. We also establish various properties of the corresponding Hankel matrices including total positivity and introduce noncommutative binomial coefficients using the LDU-factorizations of such matrices.

Travis Schedler

Multiplicative preprojective algebras are 2-Calabi–Yau

Abstract: I will explain joint work with Dan Kaplan studying the titular algebras of Crawley–Boevey and Shaw. Their moduli of representations are multiplicative analogues of Nakajima quiver varieties. When the quiver contains a cycle, we employ reduction systems (resembling Gröbner bases in a localised setting) to show that their module structure over a localised polynomial ring k[t,1/(t+q)] is that of a free product. This implies that the cotangent sequence of Cuntz and Quillen produces a two-term, self-dual bimodule resolution, and the algebras are 2-Calabi–Yau. As applications, we show that all multiplicative quiver varieties are normal with rational Gorenstein singularities, and in fact have symplectic singularities in the sense of Beauville. Also, in the case of a cycle, these algebras are noncommutative crepant resolutions of type A surface singularities. Finally, we show that the dg version of these algebras, which arise in Fukaya categories of Weinstein 4-folds, are formal.

Stanislav Shkarin

Possible Hilbert series

According to an old result, a finitely generated degree-graded algebra A satisfying $\dim A_n \leqslant n$ for some positive integer n has linear growth. More specifically, the sequence $(\dim A_m)$ is eventually periodic. A slightly stronger modification of this result was obtained by N. Iyudu [1994 Moscow Univ. Math. Bull.] We characterize all possible sequences $(\dim A_n, \dim A_{n+1}, \dots)$ in the case $\dim A_n \leqslant 3$ and $n \geqslant 3$. We use this result to list all Hilbert series of Koszul algebras with up to 3 generators.

Borya Shoikhet

The twisted tensor product of dg categories and a contractible 2-operad

We define a twisted tensor product of two small dg categories defined over a field k, and discuss the adjunction it fulfils. We prove that, provided two dg categories C and D are cofibrant, their twisted tensor product is quasi-equivalent to the ordinary tensor product. After that, we construct, by means of the twisted tensor product, a 2-operad acting on the category $Cat_dg(k)$ of small dg categories. The above mentioned property implies that this

2-operad is contractible. By a general results of Batanin, it makes the category$Cat_dg(k)$ a weak 2-category.

Sue Sierra

The Poisson spectrum of the symmetric algebra of the Virasoro algebra

Let W be the Witt algebra of vector fields on the punctured plane, and let Vir be its unique nontrivial central extension by z, the Virasoro algebra. We discuss work in progress with Alexey Petukhov to analyse Poisson ideals of the symmetric algebra of Vir.

We focus on understanding maximal Poisson ideals, which can be given as the Poisson cores of maximal ideals of Sym(Vir) and of Sym(W). We give a complete classification of maximal ideals of Sym(W) which have nontrivial Poisson cores. We then lift this classification to Sym(Vir), and use it to show that if $\lambda \neq 0$, then $(z - \lambda)$ is a maximal Poisson ideal of Sym(Vir).

Michael Wemyss

Contraction algebras, plumbings and flops

I will construct geometric models, on both the A-side and B-side of mirror symmetry, that are controlled by potentials on the two-cycle quiver. The cohomology of objects in the underlying categories are naturally modules for the associated contraction algebra, and I will explain how to use this information to obtain otherwise tricky results, such as a classification of spherical (and more generally, fat-spherical) objects. This has purely topological corollaries. One feature, which I will probably gloss over but is actually fundamental, is that our the models have a dependence on the characteristic of the ground field. This is joint work with Ivan Smith.

Oksana Yakimova

Symmetrisation and the Feigin–Frenkel centre

Let G be a complex reductive group, set $\mathfrak{g} = \operatorname{Lie} G$. The algebra $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ of *symmetric* $\mathfrak{g}$-invariants and the centre $\mathcal{Z}(\mathfrak{g})$ of the enveloping algebra $\mathcal{U}(\mathfrak{g})$ are polynomial algebras with $\operatorname{rk}\mathfrak{g}$ generators. There are several isomorphisms between them, including the symmetrisation map ϖ, which exists also in the infinite dimensional case.

The enveloping algebra $\mathcal{U}(t^{-1}\mathfrak{g}[t^{-1}])$ contains a remarkable commutative subalgebra $\mathfrak{z}(\widehat{\mathfrak{g}})$, which is closely related to the centre of the completed enveloping algebra $\widetilde{\mathcal{U}}_\kappa(\widehat{\mathfrak{g}})$ at the critical level $\kappa = -h^\vee$. By a theorem of Feigin and Frenkel, $\mathfrak{z}(\widehat{\mathfrak{g}})$ is a polynomial algebra in infinite number of generators. From 1982 until 2006, this algebra existed as an intriguing black box with many applications. Then explicit formulas for its elements appeared first in type A, later in all other classical types, and it was discovered that $\mathfrak{z}(\widehat{\mathfrak{g}})$ is a centraliser of the quadratic Casimir element.

We will discuss the type-free role of the symmetrisation map in the description of $\mathfrak{z}(\widehat{\mathfrak{g}})$ and present new explicit formulas for its generators in types B, C, D, and G_2. One of our main technical tools is a certain map $\mathsf{m}\colon \mathcal{S}^k(\mathfrak{g}) \to \Lambda^2\mathfrak{g} \otimes \mathcal{S}^{k-3}(\mathfrak{g})$.

Efim Zelmanov

Growth Functions

We will discuss growth functions of algebras and monoids.