Poincaré–Birkhoff–Witt theorems

The Antwerp Algebra Colloquium

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**Theorem (Poincaré–Birkhoff–Witt theorem)**

*Over a field of characteristic zero, the vector space underlying the universal enveloping algebra $U(\mathfrak{g})^\#$ is isomorphic to the symmetric algebra $S(\mathfrak{g}^\#)$, naturally with respect to Lie algebra maps.*
Motivation and methods

Theorem (Poincaré–Birkhoff–Witt theorem)

Over a field of characteristic zero, the vector space underlying the universal enveloping algebra \( U(\mathfrak{g})^\# \) is isomorphic to the symmetric algebra \( S(\mathfrak{g}^\#) \), naturally with respect to Lie algebra maps.

This is an important result in areas such as:

- representation theory,
- homological algebra,
- deformation theory and quantization.
Motivation and methods

There are many results that fall within the “PBW” umbrella [SW15], which:

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- associate a graded algebra to a non-graded algebra.
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- exhibit an algebra as a deformation of another, ‘nicer’ algebra,
- give us a nice basis of normal monomials for an algebra,
- associate a graded algebra to a non-graded algebra.

We were motivated by the need to produce a formal framework to state and prove such theorems for more general classes of algebras.
Motivation and methods

*Precursor:* work of Mikhalev and Shestakov [MS14] on varieties of algebras. We were motivated by the need to make precise certain intuitive ideas in their work.
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Any functor $\mathcal{S}\text{Alg} \rightarrow \mathcal{T}\text{Alg}$ that ‘only changes operations’ has a left adjoint $A \mapsto U_\mathcal{S}(A)$, so we can attempt to state what a ‘PBW-type’ theorem is in this case.
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What does it mean that this functor ‘does not depend on $A$’, exactly?

For us, this means there is an endofunctor $X$ such that $U_\mathcal{S}(A)^\#$ is isomorphic to $X(A^\#)$, naturally with respect to $\mathcal{T}$-algebra maps.
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1. We focus on functors $\psi^*$ that arise by pulling back through a morphism of operads $\psi : P \rightarrow Q$ on some category that is nice enough.

2. The universal enveloping algebra functor is the pushforward $\psi_!$, given by taking a ‘relative tensor product over $P$’.

3. Our main result shows that this functor is naturally isomorphic to some constant endofunctor on the underlying category if and only if $Q$ is a free right $P$-module.
Consequences

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- It is fully intrinsic and functorial, and unravels the ‘mystery’ behind the natural question: what property of the pair (Lie, Ass) makes the PBW theorem work?
- It gives certificates in case the PBW property fails, in the form of homology classes, which can be effectively computed.
- Explains why one cannot expect certain type of results in positive characteristic.
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- Explains why one cannot expect certain type of results in positive characteristic.
The Poincaré–Birkhoff–Witt property

Definition

An operad is a Schur endofunctor $P : C \to C$ endowed with an associative composition law $P \circ P \to P$ and a unit $\eta : 1 \to P$ for it.
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We were mostly interested in the case when $C = \text{Vect}$ so $P$ takes the form

$$V \longmapsto \bigoplus_{j \geq 0} P(j) \otimes_{S_j} V^\otimes j$$

for some sequence of representations of the symmetric groups. In this language, elements of the right are ‘where we can operate on $V$’. 

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Monads define algebras

**Definition**

An algebra over an operad is an object $c \in C$ along with a structure map $\gamma_c : P(c) \to c$ such that $\gamma_c \mu_c = \gamma_c P(\gamma_c)$ and $\gamma_c \eta_c = 1_c$. Thus, $P(c) \to c$ consists of many maps $P(j) \otimes S_j V \otimes j \to V$, so each $x \in P(j)$ defines an equivariant operation $x : V \otimes j \to V$. For example, if $C(j)$ is the trivial representation, then we are considering multi-linear symmetric maps $S_j(V) \to V$. Defining the composition law in the only way possible defines the commutative operad, $\text{Com}$.
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If $\psi : P \to Q$ is a morphism of operads, the pullback functor $\psi^*$ sends a $Q$-algebra $(c, \gamma_c)$ to the algebra $(c, \gamma_c \psi_c)$. 
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The left adjoint $\psi_!$ of $\psi^*$ is called the \textit{universal enveloping algebra functor of $\psi$}. It exists under mild assumptions on $C$. 
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**Definition**

If \( \psi : P \to Q \) is a morphism of operads, the pullback functor \( \psi^* \) sends a \( Q \)-algebra \((c, \gamma_c)\) to the algebra \((c, \gamma_c \psi_c)\).

The left adjoint \( \psi_! \) of \( \psi^* \) is called the *universal enveloping algebra functor of \( \psi \).* It exists under mild assumptions on \( C \).

For example, if \( \psi_V : \mathbb{L}(V) \to T(V) \) is the natural inclusion of the free Lie algebra in the tensor algebra, \( \psi_!(\mathfrak{g}) \) is precisely \( U(\mathfrak{g}) \).
How to control this procedure?

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A right module over an operad $P$ is another endofunctor $M$ along with a map $M \circ P \rightarrow P$ compatible with the composition law of $P$. 
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**Definition**

A right module over an operad $P$ is another endofunctor $M$ along with a map $M \circ P \to P$ compatible with the composition law of $P$.

A right module is *free* if it is isomorphic to one of the form $X \circ P$ where $X$ is an endofunctor and the structure map is $1_X \circ \mu : X \circ F \circ F \to X \circ F$. 
Recall we say the PBW property holds if there is an endofunctor $X$ such that $\psi_!(A)^\#$ is isomorphic to $X(A^\#)$, naturally with respect to algebra maps.
The main result

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Theorem (Dotsenko–T. ’18)

A map of operads\footnote{More generally, monads} $\psi : P \to Q$ has the PBW property if and only if it makes $Q$ into a free right $P$-module. In this case, the desired functor $X$ is any free right basis of $Q$. 
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Like the innocent method of counting in two ways, in this case the proof of the result does not matter as much as the fact we have many powerful tools to address whether an object is free or not: this is, more or less, the reason homological algebra exists!

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1More generally, monads

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Poincaré–Birkhoff–Witt theorems
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Applications

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**Theorem**

Let $P$ be a weight graded operad, $M$ a weight graded right $P$-module. Then:

$$M \text{ is free if and only if } H_+(B(M, P, 1)) = 0.$$ 

In such case, it is free with basis $X = H_0(B(M, P, 1))$. 

Main takeaway: we do not have to guess a Poincaré–Birkhoff–Witt theorem, but rather compute something. The output of that result will tell us what the answer is.
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Classical PBW theorem

There is a map of operads $\text{Lie} \rightarrow \text{Ass}$ sending $[x_1, x_2] \mapsto x_1x_2 - x_2x_1$.

Theorem (Poincaré–Birkhoff–Witt)

The associative operad is free as a right module over the Lie operad with basis given by the endofunctor $V \mapsto S(V)$. 

Proof. Filter the associative operad using the number of Lie brackets an operation uses (polarize the associative product). The associated graded module is exactly the operad controlling Poisson algebras. As a right module, it is $\text{Com} \circ \text{Lie}$, so it is free. By a spectral sequence argument, the associative operad is free with the same basis. □
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A pre-Lie PBW theorem

*Pre-Lie algebras:* defined by a single operation $x_1 \circ x_2$ whose associator is symmetric in the last two variables. This implies $[x_1, x_2] = x_1 \circ x_2 - x_2 \circ x_1$ is a Lie bracket.
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**Theorem (Dotsenko–T. ’18)**

*The pre-Lie operad is free as a right module over the Lie operad with basis given by the endofunctor $V \mapsto R(S(V))$.***
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Theorem (Dotsenko–T. ’18)

The pre-Lie operad is free as a right module over the Lie operad with basis given by the endofunctor $V \mapsto R(S(V))$.

Here $R$ is the endofunctor of rooted trees for which no vertex has exactly one child.
A pre-Lie PBW theorem

Proof.

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2. By a result of V. Dotsenko the associated graded module is exactly the operad controlling the $F$-manifold algebras of C. Hertling and Yu. I. Manin.
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2. By a result of V. Dotsenko the associated graded module is exactly the operad controlling the $F$-manifold algebras of C. Hertling and Yu. I. Manin.
3. It can be shown it has a basis of tree monomials that is preserved under the action of the Lie operad, so it is free on some sub-basis of it.
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4. By a spectral sequence argument, the pre-Lie operad is free with the same basis.
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\[ \square \]

*Basis:* by work of Dotsenko–Flynn-Connolly using the Koszul complex $K(\text{PreLie}, \text{Lie}, 1)$ to compute bar homology.
A question of J.-L. Loday

*Dendriform algebra*: a vector space $V$ endowed with two operations $x_1 \prec x_2$ and $x_1 \succ x_2$ plus three identities.

\[
(x_1 \prec x_2) \prec x_3 - x_1 \prec (x_2 \prec x_3) = x_1 \prec (x_2 \succ x_3),
\]
\[
(x_1 \succ x_2) \succ x_3 - x_1 \succ (x_2 \succ x_3) = -(x \prec x_2) \succ x_3
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▶ The operation $x_1 \circ x_2 = x_1 \prec x_2 - x_2 \succ x_1$ is pre-Lie.

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- We obtain a morphism $\text{PreLie} \rightarrow \text{Dend}.$

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**Theorem (Dotsenko–T. ’18)**

*The map $\text{PreLie} \rightarrow \text{Dend}$ has the PBW property.*
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Proof.

1. Filter the dendriform operad using the number of pre-Lie brackets an operation uses (polarize the products $≺, ⊼$).
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2 The associated graded module is exactly the operad controlling the pre-Poisson algebras of M. Aguiar.
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Further directions

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One of the upshots of having written our work in the language of monads and modules is that we can extrapolate it to more complex settings.

_Differential graded objects_. A natural direction to move towards is that of ‘derived’ results: what happens if we allow our objects to be differential graded or allow for homotopy algebras?

_Operads as algebras_. We can also produce an interesting feedback loop if we consider coloured operads: every operad can be made into a pre-Lie algebra, so there is an enveloping operad functor whose inputs are pre-Lie algebras.
Envelopes of homotopy algebras

We can allow $\alpha : P \to Q$ to be a morphism of dg operads. What does it mean to have a PBW property here?

Definition (Khoroshkin–T. '19)
We say $\alpha$ is derived PBW if the natural transformation $H(\alpha)! \to H(\alpha(V))$ is a natural isomorphism for every $P$-algebra $V$.

As the definition shows, the idea is to obtain control on the homology of universal envelopes of 'complicated' algebras through the non-dg envelope $H(\alpha)!HV$.
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Almost-freeness

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The main result we obtained with A. Khoroshkin [KT20] is the following:

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Almost-freeness implies dPBW

Theorem (Khoroshkin–T. ’19)

If $\alpha : P \longrightarrow Q$ makes $Q$ into an almost free right $P$-module with basis of cycles $X$, then $\alpha$ is derived PBW and we have a natural isomorphism

$$H(\alpha_!(V)) \longrightarrow X(HV).$$

Corollary: the $A_\infty$ operad is almost free over the $L_\infty$ operad by techniques coming from homological perturbation theory, producing a unified approach to previous work of V. Baranovsky and J. Moreno-Fernández.
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Some consequences

Idea: there is a filtration on $\text{Ass}_\infty$ for which $\text{gr} \text{ Ass}_\infty = \text{Poiss}_\infty$ and that $\text{Poiss}_\infty$ is chain homotopy equivalent to $\text{Com} \circ \text{Lie}_\infty$.

- The homology groups of $U(\mathfrak{g})$ for any $L_\infty$-algebra $\mathfrak{g}$ are given by $S(H(\mathfrak{g}))$. 
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- The models of Baranovsky and of Moreno–Fernández are \(A_\infty\)-isomorphic and \(A_\infty\)-quasi-isomorphic to the universal envelope of Lada–Markl.
There is a map $\text{PreLie}_\mathbb{N} \to \text{nsOp}$ from the operad controlling weight-graded pre-Lie algebras to the operad controlling ns-operads. An open question is

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R. Campos and P. Tamaroff, *Differential forms on smooth operadic algebras*.


