The Tamarkin–Tsygan calculus of an algebra

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Motivation and origins: the Cartan calculus

For a smooth manifold M, the spaces $\Omega(M)$ of forms on M and $\Theta(M)$ of polyvector fields on M are endowed with a Cartan calculus, that satisfies the following formulas:

$$L_X = [d, i_X], \quad i_{X \wedge Y} = i_X \cdot i_Y$$

$$L_{\{X,Y\}} = [L_X, L_Y], \quad [i_X, L_Y] = i_{\{X,Y\}}$$

$$L_{X \wedge Y} = L_X \cdot i_Y + (-1)^{|X|} i_X \cdot L_Y, \quad [i_X, i_Y] = 0$$

Here i is contraction, L is the Lie derivative, $\{-,-\}$ the Schouten bracket and the products are wedge products, while d is the de Rham differential.

Motivation and origins: the HKR theorem

For a smooth commutative algebra A, we know from the HKR theorem that we have identifications of *algebras*

$$\mathrm{HH}_*(A) = \Lambda_A^* \Omega_A^1, \quad \mathrm{HH}^*(A) = \Lambda_A^* \mathrm{Der}(A).$$

If we consider $A = \Bbbk[X]$ the coordinate ring of a smooth variety X, then $\mathrm{Der}(A)$ is the space of vector fields on X and Ω_A is that of forms on X.

Then the above identification gives us a "Cartan calculus": a wedge product on fields, a contraction of forms with fields, a de Rham differential on forms, and a Lie bracket on fields.

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The non-commutative analogue

We can produce an analogous picture when A is an arbitrary associative algebra, the *Tamarkin–Tsygan calculus of A*.

It consists of the *cup product* and the *Gerstenhaber bracket* on $\mathrm{HH}^*(A)$, the *cap product action* of $\mathrm{HH}^*(A)$ on $\mathrm{HH}_*(A)$ and the boundary map d on $\mathrm{HH}_*(A)$, the *differential of Connes–Tsygan.*

In fact, one can define the previous operators on the *chain level*, that is, on the usual complexes $C^*(A)$ and $C_*(A)$ that compute Hochschild cohomology and homology.

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An intrinsic definition?

The above produces an assignment from associative algebras to Tamarkin–Tsygan calculi, which are algebras over a 2-coloured operad.

From the work of J. Stasheff, we can deduce that the bracket is intrinsic to the homotopy category of dg algebras: we can compute it as the Lie bracket on derivations on any quasi-free model of our algebra.

Question. Can we do the same with the whole Tamarkin–Tsygan calculus, that is, produce from the homotopy type of A a datum that gives this calculus?

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A simple homological lemma

Lemma. Let C be a right half-plane double complex and assume that for every $q \in \mathbb{N}_0$, $H_{\geqslant 2}(C_{*,q}, d_{\mathrm{hor}}) = 0$. Let $f : C_{*\geqslant 2,*} \longrightarrow C_{*\leqslant 1,*}[1,0]$ be the map induced by the horizontal differential $d_{*,2}$. Then $\mathrm{Tot}(C)$ is naturally quasi-isomorphic to the totalization of the cokernel of f.

Proof. Un dessin, the fact that the totalization of the cone of f is equal to the totalization of C, and that $\mathrm{Tor}(\ker f)$ is acyclic. ◀

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The lemma for Hochschild (co)homology

Lemma. Suppose that $B \longrightarrow A$ is a model of A, and let

$$\overline{\Omega}_B = \mathrm{coker}(C_2(B) \longrightarrow C_1(B)), \quad \mathrm{Der}(B) = \ker(C^1(B) \longrightarrow C^2(B))$$

the kernel and cokernel of the *external* Hochschild boundaries on $C_*(B)$ and $C^*(B)$, respectively. We can compute (co)homology of A as follows:

$$\mathrm{HH}_*(A) = H_*(\mathrm{cone}(\overline{\Omega}_B \longrightarrow B)), \quad \mathrm{HH}^*(A) = H^*(\mathrm{cone}(B \longrightarrow \mathrm{Der}(B)).$$

Note. The maps being "coned" are induced by $d_1 : C_1(B) \longrightarrow C_0(B)$ and by $d^0 : C^0(B) \longrightarrow C^1(B)$, the commutator and the adjoint.

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Fields and forms

Theorem. Let A be an associative algebra and $(TV, d) = B \longrightarrow A$ a quasi-free model of A. There are complexes

$$V^B = B \oplus \mathrm{Der}(B)[-1], \quad V_B = B \oplus \overline{\Omega}_B[1]$$

of fields and forms that compute Hochschild cohomology and Hochschild homology of A and depend only on the homotopy type of A in Alg.

Proof. A corollary of the last lemma and the fact the (co)homology of free algebras vanishes in degrees above 1 or, equivalently, that operadic (co)homology of free algebras vanishes in positive degrees. ◀

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Simplifying the spaces of fields and forms

The cone of the adjoint map $B \longrightarrow \mathrm{Der}(B) := D_B$ and that of the commutator $\overline{\Omega}_B \longrightarrow B$ produce for us fields and forms. Since B is quasi-free, we can present V^B and V_B in simpler terms.

Lemma. For $B = TV$ a model of A, there are natural isomorphisms $D_B \to \hom(V, B)$ and $\overline{\Omega}_B \to B \otimes V$.

Proof. All follows from universal properties of TV. ◀

Note. Since $H(B) = A$, the complexes D_B and $\overline{\Omega}_B$ compute Hochschild cohomology and homology in degrees above one. In view of the lemma, we will write fields as $X = \lambda + f$ and forms as $\omega = b + b' dv$, where $dv = [v]$ is a bar element in bar degree 1.

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The Gerstenhaber bracket

Theorem. (Stasheff) The Gerstenhaber bracket is precisely the Lie bracket on $\mathrm{hom}(\mathsf{B}A, A) = \mathrm{Coder}(\mathsf{B}A)$.

We can use this result to deduce the following generalization:

Theorem. If $B \longrightarrow A$ is a quasi-free model, then the dg Lie algebra $\mathrm{cone}(\mathrm{ad}_B)$ is of the homotopy type of the Lie algebra $\mathrm{HH}^*(A)$ where $[-,-]$ is the Gerstenhaber bracket.

Proof. Stasheff's work proves it for $B = \Omega\mathsf{B}A$, and given another model $B \longrightarrow A$, we can factor this through $\Omega\mathsf{B}A \longrightarrow A$. ◀

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The cap product

Note that the mapping cones we have constructed in terms of B are just the usual complexes computing Hochschild (co)homology

$$0 \longrightarrow TV \otimes V \longrightarrow TV \longrightarrow 0$$

$$0 \longrightarrow TV \longrightarrow \hom(V, TV) \longrightarrow 0$$

that compute Hochschild homology and cohomology of the algebra $B^{\#}$. To get that of B we just introduce the internal differential of it.

These two complexes are paired in a usual way: maps $V \longrightarrow TV$ act on forms $TV \otimes V$ on the second coordinate, and scalars TV act on forms on the first and second coordinates.

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Theorem. The usual pairing between the two complexes above induces the cap product action when truncated using the homological lemma. Concretely, we have that

$$\omega(X) = b\lambda + (-1)^{|b'||f|} b' f v + (-1)^{|\lambda|(|v|+1)} b' \lambda dv$$

for a field $X = \lambda + f$ and a form $\omega = b + b' dv$.

Proof. One checks that $\partial(\omega X) = (\partial\omega)(X) + (-1)^{|\omega|}\omega(\partial X)$ holds for forms and fields, meaning the pairing is compatible with the internal differentials coming from B. ◀

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The differential of Connes–Tsygan

Intermezzo. (Feigin–Tsygan) One can compute cyclic homology of A by totalizing the infinite 2-periodic complex $B \longleftarrow \overline{\Omega}_B \longleftarrow B \longleftarrow \cdots$ where the map $B \longrightarrow \overline{\Omega}_B$ is the projection of the universal derivation.

Note. In positive characteristic we can simply use the complex $B/[B,B]$ to compute cycic homology. This is the target of the universal invariant bilinear form $\mathrm{tr} : B \longrightarrow B/[B,B]$.

Proposition. The ISB sequence is obtained by including $\overline{\Omega_B} \longrightarrow B$ as the first two columns and projecting. In this case, the connecting morphism sends a form $\omega = b + b'dv$ to $d\omega = db$.

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Enter cyclic actions on trees

There is a Leibniz rule on forms, $a[bc] = ab[c] + (-1)^{|c|(|a|+|b|)}ca[b]$, since they are obtained as the cokernel of the Hochschild boundary. This gives us the following simple description of the differential of Connes:

Theorem. For a form $\omega = b + b'dv$ with $b = v_0 v_1 \cdots v_n$, we have that

$$d\omega = \sum_{i=0}^{n} (-1)^{\varepsilon} v_{i+1} \cdots v_n v_0 \cdots v_{i-1} dv_i.$$

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The final result

All that we have done, plus the fact the cup product is obtained from the quadratic part of $\partial : B \longrightarrow B$, gives us the desired final result:

Theorem. The Tamarkin–Tsygan calculus of an algebra A can be computed using the datum $(V^B, V_B, \smile, \frown, [-,-], d)$ obtained from a model $B \longrightarrow A$ as before, and it descends to a well defined functor $\mathrm{Ho}(\mathrm{Alg})^\times \longrightarrow \text{Calc-Alg}$.

Question. We can obtain the derived category of A through B. Is it possible to upgrade this to derived invariance and not just homotopy invariance?

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The story for (cyclic) operads

Fix an operad $\mathcal{O}$ and $\mathcal{O}$-algebra A along with a cofibrant replacement $B \longrightarrow A$. Then we can compute operadic (André–Quillen) cohomology and homology of A as

$$H^*_{\mathcal{O}}(A) = \mathbb{R}\,\mathrm{Der}_{\mathcal{O}}(A, A) = H^*(\mathrm{Der}_{\mathcal{O}}(B, B)) \text{ and,}$$

$$H_*^{\mathcal{O}}(A) = \mathbb{L}(A \otimes_U \Omega_{\mathcal{O},A}) = H_*(B \otimes_U \Omega_{\mathcal{O},B}),$$

so nothing stops us from attempting to go through the process above which was the case $\mathcal{O} = \mathrm{As}$.

What works and what is missing

- André–Quillen cohomology and homology, by its very definition, vanishes on positives degrees for free algebras, so our homological lemma works.
- For B quasi-free over V, the identification $\mathrm{Der}\, B = \hom(V, B)$ is obvious, and $B \otimes_U \Omega_{\mathcal{O},B} = B \otimes V$ follows by carefully tracing universal properties.
- The Lie bracket and the cap product action exist by virtue of the same arguments. We obtain the a cup product from the quadratic part of the differential of B.
- When $\mathcal{O}$ is cyclic, the norm element of the cyclic action should give the circle action and then the differential of Connes, but this may be delicate since the ISB sequence of Getzler–Kapranov is different from the classical one.

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Thank you!

Preprint: `https://maths.tcd.ie/~pedro/TTCpreprint.pdf`