Universidad de Buenos Aires
Facultad de Ciencias Exactas y Naturales
Departamento de Matemática

Tesis de Licenciatura

Cohomología de Especies Combinatorias

Pedro Tamaroff

Director: Mariano Suárez-Álvarez

Fecha de Presentación: 31 de Marzo de 2017

Contents

Introducción 1

Chapter I. Combinatorial species 5
1. The category of species 5
2. Basic definitions 6
3. Examples 8

Chapter II. Monoidal categories and simplicial homology 11
1. Monoidal categories 11
2. Linear and abelian categories 15
3. The Cauchy monoidal structure on species 17
4. Monoids, comonoids and bimonoids in species 20
5. Hopf monoids in the category of species 24
6. Monoidal structure on the category of bicomodules 27
7. The simplicial category 28
8. Cohomology in abelian monoidal categories 31

Chapter III. The cohomology of combinatorial species 35
1. Definitions and first examples 35
2. First computations 39
3. The cup product 41
4. Antisymmetrization of cochains 43
5. The cohomology ring of the species of linear orders 45
6. The cohomology ring of the species of compositions 54
7. Determined and representable species 60

Chapter IV. The combinatorial complex 73
1. The spectral sequence 75
2. Computation of the E_1-page 78
3. Convergence of the spectral sequence 84
4. The differential of the combinatorial complex 87
5. Some calculations using the combinatorial complex 92

6. Multiplicative structure of the spectral sequence 96

Chapter V. Future work and problems 101

Appendix A. Homological algebra 107
1. Bifunctors and homology 107
2. Spectral sequences 108
3. Inverse limits and the Mittag-Leffler condition 110

Bibliography 115

A mi familia, mis amigos, y profesores, que hicieron posible este trabajo.

Introducción

En su artículo [Joy1981], André Joyal presentó la noción de especie combinatoria, con el objetivo de categorificar la idea de función generadora. En líneas generales, una especie combinatoria $\mathbf{x}$ consiste de una sucesión de conjuntos $\{\mathbf{x}(n)\}_{n\geqslant 0}$, cada uno de ellos dotado de una acción del grupo simétrico correspondiente, que codifica la forma en que las estructuras combinatorias se "reetiquetan". Por ejemplo, la especie de árboles A tiene como $A(n)$ al conjunto de árboles con vértices etiquetados biyectivamente por los elementos del conjunto $\{1,\ldots,n\}$, sobre los que el grupo S_n actúa de la forma evidente.

Recientemente, Aguiar y Mahajan, en el contexto del estudio de las especies combinatorias con el objeto de construir álgebras de Hopf, se vieron llevados a considerar variaciones o deformaciones de la estructura monoidal de Cauchy de la categoría de especies, asi como de sus monoides y comonoides. Esto lleva naturalmente a la consideración de una teoría de cohomología para especies: de esta manera, a cada comonoide $\mathbf{x}$ asignan grupos de cohomología $H^*(\mathbf{x},\mathbb{Z})$, y su interés se centra en el grupo $H^2(\mathbf{x},\mathbb{Z})$, que parametriza las deformaciones de la estructura de comonoide de $\mathbf{x}$.

En esta tesis estudiamos en detalle la construcción de esta teoría de cohomología y damos los primeros ejemplos de su cálculo. Siguiendo la presentación de [AM2010], consideramos con cierto cuidado la estructura monoidal de la categoría de especies y, en particular, las representaciones y correpresentaciones de la especie exponencial E. Esta especie E juega un rol especial en toda la teoría, ya que la teoría de cohomología de Aguiar–Mahajan puede verse como una especializacion de los funtores Ext en la categoría de los E-bicomódulos. Esto nos permite encarar su estudio con las herramientas del álgebra homológica.

Nuestro trabajo estuvo organizado alrededor del cálculo de la cohomología de ejemplos concretos. En particular, centramos inicialmente nuestra atención sobre la especie de órdenes lineales L: el problema de la determinación de su cohomología está ya planteado en la monografía de Aguiar–Mahajan y resulta, de hecho, no trivial. Lo resolvemos estableciendo una conexión con la geometría del arreglo de trenzas y el correspondiente complejo de Coxeter. Hecho esto, consideramos algunos ejemplos más —especies asociadas a complejos simpliciales, y otros monoides de Hopf importantes en la teoría, como la especie de composiciones Σ y la especies de particiones Π.

Analizando el patrón de cálculo en estos ejemplos, pudimos abstraer un procedimiento general. El resultado final de esto es una descripción alternativa de la cohomología de una especie: mostramos que a partir de cada especie $\mathbf{x}$ se puede construir un complejo $CC^*(\mathbf{x})$, que llamamos el "complejo combinatorio" de $\mathbf{x}$, cuya cohomología coincide, en los casos favorables, con la cohomología de la especie. Esta construcción proviene de considerar sobre la especie $\mathbf{x}$ una filtración natural por cardinalidad y, a partir de ésta, construir una sucesion espectral que converge a $H^*(\mathbf{x})$ y que estudiamos en detalle. Esto es enteramente análogo a la forma en que se puede describir la cohomología de un CW-complejo en términos de su cohomología celular. Más aún, cuando $C^*(\mathbf{x})$ admite un producto cup, esa sucesión espectral es de álgebras y esto permite describir explícitamente el producto cup de $H^*(\mathbf{x})$ en términos del complejo combinatorio de $\mathbf{x}$.

Las ventajas del complejo $CC^*(\mathbf{x})$, respecto al complejo canónico $C^*(\mathbf{x})$ son muchas: en primer lugar, cada componente de $CC^*(\mathbf{x})$ es un k-modulo finitamente generado si $\mathbf{x}$ es finita en cada cardinal. En contraste, el complejo canónico que calcula $H^*(\mathbf{x})$ no es, excepto es casos triviales, localmente finito. En segundo lugar, el diferencial del complejo $CC^*(\mathbf{x})$ en grado q depende sólo de las estructuras combinatorias en cardinales q y $q+1$, e involucra —en un sentido estricto— la menor cantidad de información posible de la estructura de bicomódulo de $\mathbf{x}$. De forma completamente opuesta, el diferencial del complejo canónico en grado q involucra a todas las estructuras en cardinales menores o iguales a $q+1$ y toda la información de la estructura de bicomódulo de $\mathbf{x}$.

El cálculo de la cohomología usando esta descripción alternativa es realmente más simple en la práctica. Por ejemplo, pudimos recuperar todos los hechos en el Capítulo III de manera directa de una forma mucho más rápida y sencilla. Por otro lado, es importante notar que esta descripción de la cohomología, a diferencia de la original, se presta a ser implementada en una computadora: por ejemplo, a pesar de que no conocemos el álgebra de cohomología de la especie de grafos simples —cuya determinación completa es probablemente un problema muy difícil— podemos calcularla explícitamente en grados bajos. En particular„ esta teoría transforma la determinación del segundo grupo de cohomología de una especie “razonable” en un problema computacionalmente muy sencillo.

La tesis está organizada en cinco capítulos y un apéndice.

En el capítulo I recordamos las nociones básicas sobre especies combinatorias y presentamos los ejemplos que vamos a estudiar a lo largo de este trabajo, junto con otros que son relevantes y que motivan la teoría. En el capítulo II presentamos el lenguaje necesario para enmarcar la teoría de cohomología de especies combinatorias. Sus primeras secciones tratan sobre categorías monoidales, que usamos para dotar de las estructuras necesarias a la categoría de especies combinatorias, y su última sección presenta el método simplicial estándar para constuir teorías de cohomología vía objetos simpliciales en categorías monoidales abelianas.

En el capítulo III definimos, por fin, la teoría de cohomología para especies, especializando las construcciones generales del capítulo anterior a nuestro contexto. Estudiamos en detalle la categoría de E-bicomódulos, con E la especie exponencial, y su teoría de cohomología dada por $\operatorname{Ext}_{E^e}(?,E)$, que, como dijimos, coincide con el funtor de cohomología de especies. Para ciertas especies $\mathbf{x}$, mostramos que $H^*(\mathbf{x})$ es, de manera natural, un álgebra graduada, construyendo un “producto cup” sobre ella. El capítulo concluye con los cálculos explícitos prometidos, incluyendo en cada caso su estructura de álgebra.

En el capítulo IV probamos que a todo E-bicomodulo $\mathbf{x}$, que cumple una condición técnica (que se satisface en los ejemplos de interés) se le puede asociar un complejo de cocadenas $CC^*(\mathbf{x})$, el complejo combinatorio, que calcula la cohomología $H^*(\mathbf{x})$. En la situación en que esta cohomología es un álgebra, exhibimos sobre $CC^*(\mathbf{x})$ una estructura de álgebra diferencial graduada que induce la estructura correcta sobre $H^*(\mathbf{x})$.

La tesis finaliza en el Capítulo V, en el que planteamos algunos problemas que surgen de nuestro trabajo asi como algunos posibles caminos a seguir para continuar el estudio iniciado aquí.

Los resultados del Capítulo III son originales, con excepción de las Proposiciones 2.1, 2.2 y 2.3, ya presentes en la tesis de maestría de Javier Cóppola [**Cop2015**], mientras que los resultados del Capítulo IV son originales.

Nuestra referencia general para las herramientas del álgebra homológica es el libro de Weibel [**Wei1994**], y para las sucesiones espectrales, ese libro y el de McCleary

[**McC2001**]. Para especies, referimos al lector al artículo de Joyal [**Joy1981**] y al libro de Labelle, Leroux y Bergeron [**LBL1998**]. Finalmente, para el formalismo de las categorías monoidales, nuestra referencia es el libro de Kassel [**Kas1995**], para las categorías abelianas, es el de Freyd [**Fre1964**], y para el formalismo simplicial, son el libro [**Wei1994**] y el de MacLane [**Mac1971**].

A lo largo de toda la tesis, k denota un anillo conmutativo y unital, y cuando escribamos $\otimes$ y hom, estaremos considerando los funtores de k-modulos, salvo mención en contrario.

CHAPTER I

Combinatorial species

Most of this chapter follows the exposition in [Joy1981].

1. The category of species

Denote by $\mathsf{Set}^\times$ the category of finite sets and bijections. A *combinatorial species* over a category C is a functor $X : \mathsf{Set}^\times \longrightarrow \mathsf{C}$. Concretely, a combinatorial species X is obtained by assigning

S1. to each finite set I an object $\mathbf{x}(I)$ in C,
S2. to each bijection $\sigma : I \longrightarrow J$ an arrow $\mathbf{x}(\sigma) : \mathbf{x}(I) \longrightarrow \mathbf{x}(J)$,

in such a way that

S3. for every pair of composable bijections σ and τ, we have $\mathbf{x}(\tau\sigma) = \mathbf{x}(\tau)\mathbf{x}(\sigma)$ and,
S4. for every finite set I, it holds that $\mathbf{x}(\mathrm{id}_I) = \mathrm{id}_{\mathbf{x}(I)}$.

In particular, for every finite set I we have a map $\sigma \in \mathrm{Aut}(I) \longmapsto \mathbf{x}(\sigma) \in \mathrm{Aut}(\mathbf{x}(I))$ which gives an action of the symmetric group with letters in I on $\mathbf{x}(I)$. The category $\mathsf{Set}^\times$ is a grupoid, and it has as skeleton the full subcategory spanned by the sets[1] $[n] = \{1, \ldots, n\}$, and a species is determined, up to isomorphism, by declaring its values on the finite sets $[n]$ and on every $\sigma \in S_n$. In view of this, one can think of a combinatorial species as a sequence $(\mathbf{x}_n)_{n\geqslant 0}$ of objects in C endowed with S_n actions $(S_n \times \mathbf{x}_n \longrightarrow \mathbf{x}_n)_{n\geqslant 0}$.

We denote by $\mathsf{Sp}(\mathsf{C})$ the category $\mathsf{Fun}(\mathsf{Set}^\times, \mathsf{C})$ of species over C, whose morphisms are natural transformations: explicitly, an arrow $\eta : \mathbf{x} \longrightarrow \mathbf{y}$ is an assignment of a map $\eta_I : \mathbf{x}(I) \longrightarrow \mathbf{y}(I)$ to each finite set I, in such a way that for any bijection $I \xrightarrow{\sigma} J$ the

[1] In particular, $[0] = \varnothing$.

following diagram commutes

$$\begin{array}{ccc} \mathbf{x}(I) & \xrightarrow{\eta_I} & \mathbf{y}(I) \\ {\scriptstyle \mathbf{x}(\sigma)}\downarrow & & \downarrow{\scriptstyle \mathbf{y}(\sigma)} \\ \mathbf{x}(J) & \xrightarrow{\eta_J} & \mathbf{y}(J) \end{array}$$

This says that we must specify, for each finite set I, an $\mathrm{Aut}(I)$-equivariant map $\eta_I : \mathbf{x}(I) \longrightarrow \mathbf{y}(I)$. If we view species as sequences of objects on which the symmetric grupoid acts, a morphism of species $\mathbf{x} \longrightarrow \mathbf{y}$ is a sequence of equivariant maps $(\eta_n : \mathbf{x}_n \longrightarrow \mathbf{y}_n)_{n\geqslant 0}$.

Our main interest will lie on species over sets or vector spaces. We write Sp for the category of species over Set, the category of sets and functions, and call its objects *set species*. If a species takes values on the subcategory FinSet of finite sets we call it a *finite* set species, and if $\mathbf{x}(\varnothing)$ is a singleton, we say it is *connected*. We write Sp_k for the category of species over ${}_k\mathrm{Mod}$, the category of modules over k, and call its objects *linear species*. If a species takes values on the subcategory ${}_k\mathrm{mod}$ of finite generated modules we call it a linear species of *finite type*, and we say it is *connected* if $\mathbf{x}(\varnothing)$ is k-free of rank one.

Denote by $k[?]$ the functor $\mathsf{Set} \longrightarrow {}_k\mathrm{Mod}$ that sends a set $\mathbf{x}$ to the free k-module with basis $\mathbf{x}$, which we will denote by $k\mathbf{x}$, and call it the *linearization of* $\mathbf{x}$. By postcomposition, we obtain a functor $\mathscr{L} : \mathsf{Sp} \longrightarrow \mathsf{Sp}_k$ that sends a set species $\mathbf{x}$ to the linear species $k\mathbf{x}$. The species in Sp_k that are in the image of $k[?]$ are called *linearized species*. Thus, a linearized species $\mathbf{x} = k\mathbf{x}_0$ is such that, for every finite set I, $\mathbf{x}(I)$ has a chosen basis $\mathbf{x}_0(I)$, the morphisms $\mathbf{x}(I) \longrightarrow \mathbf{x}(J)$ map basis elements to basis elements, and the action of $\mathrm{Aut}(I)$ on $X(I)$ is by permutation of the basis elements.

It is important to note that for each non-negative integer j, there is an embedding $[?]^{(j)} : {}_{kS_j}\mathrm{Mod} \longrightarrow \mathsf{Sp}_k$ of the category of kS_j-modules in Sp_k by assigning a kS_j-module V the species $V^{(j)}$ that has $V^{(j)}(I) = V$ if I has j elements and $V^{(j)}(I) = 0$ if not. This determines $V^{(j)}$ up to a choice of bijections $\beta_I : I \longrightarrow [j]$ for each finite set I with j elements that dictate how $\mathrm{Aut}(I)$ acts on $V^{(j)}(I)$ for such finite set. In a similar fashion, there are projections $[?]_{(j)} : \mathsf{Sp}_k \longrightarrow {}_{kS_j}\mathrm{Mod}$ that assign, to each $\mathbf{x}$, the kS_j-module $\mathbf{x}_{(j)} = \mathbf{x}([j])$. It is clear that $([?]^{(j)}, [?]_{(j)})$ is an adjoint pair, and, in fact, the assignment

(1) $\qquad K : \mathbf{x} \in \mathsf{Sp} \longrightarrow (\mathbf{x}([n]))_{n\geqslant 0} \in \mathsf{R}$

is an equivalence of categories, where R is the direct product of the categories $\mathsf{R}_j = {}_{S_j}\mathrm{Mod}$ for $j \geqslant 0$. This also applies to linear species, of course.

2. Basic definitions

Given a species $\mathbf{x} : \mathsf{Set}^\times \longrightarrow \mathsf{Set}$ and a finite set I, we call $\mathbf{x}(I)$ the *set of structures of species* $\mathbf{x}$ *over* I. If $s \in \mathbf{x}(I)$, we call I the *underlying set* of s, and call s an *element* of $\mathbf{x}$ or an $\mathbf{x}$-*structure.* If $I \xrightarrow{\sigma} J$ is a bijection, the element $\mathbf{x}(\sigma)(s) = t$ is the structure over J obtained by *transporting* s *along* σ, which we will usually denote, for simplicity, by σs. Two $\mathbf{x}$ structures s and t over respective sets I and J are said to be *isomorphic* if there is a bijection $\sigma : I \longrightarrow J$ that transports s to t, and we say σ is a *structure isomorphism* from s to t. A permutation that transports a structure s to itself is said to be an *automorphism* of s.

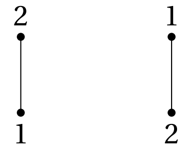
In most cases, if $\mathbf{x}$ is a species and I is a set, $\mathbf{x}(I)$ consists of a collection of combinatorial structures of some kind labelled in some way by the elements of I. For example, there is a species Pos that assigns to every finite set I the set $\mathsf{Pos}(I)$ of partial orders on I, and to every bijection $\sigma : I \longrightarrow J$ the function $\mathsf{Pos}(\sigma) : \mathsf{Pos}(I) \longrightarrow \mathsf{Pos}(J)$ which assigns to every order on I the unique order on J that makes σ an order isomorphism: in concrete terms, $\mathsf{Pos}(\sigma)$ "relabels" a poset on I according to σ.

We denote by $\mathrm{el}(\mathbf{x})$ the *category of elements of* $\mathbf{x}$: the objects of this category are the $\mathbf{x}$-structures, and its arrows are structure isomorphisms. This category is a groupoid, and there is a forgetful functor $u_{\mathbf{x}} : \mathrm{el}(\mathbf{x}) \longrightarrow \mathsf{Set}$ that sends a structure to its underlying set. In this way, we obtain a functor $\mathrm{el} : \mathsf{Sp} \longrightarrow \mathsf{Grpd}$ to the category of grupoids: if $\eta : \mathbf{x} \longrightarrow \mathbf{y}$ is a morphisms of species, $\mathrm{el}(\eta) : \mathrm{el}(\mathbf{x}) \longrightarrow \mathrm{el}(\mathbf{y})$ is the functor that sends an $\mathbf{x}$-structure s on a set I to the $\mathbf{y}$-structure $\eta_I(s)$ on I. Moreover, this renders the following triangle commutative

$$\begin{tikzcd}
\mathrm{el}(\mathbf{x}) \arrow[rr, "\mathrm{el}(\eta)"] \arrow[dr, "u_{\mathbf{x}}"'] & & \mathrm{el}(\mathbf{y}) \arrow[dl, "u_{\mathbf{y}}"] \\
& \mathsf{Set} &
\end{tikzcd}$$

There is partial converse to this construction: if one has a functor $\rho : \mathrm{el}(\mathbf{x}) \longrightarrow \mathrm{el}(\mathbf{y})$ such that $u_{\mathbf{y}}\rho = u_{\mathbf{x}}$, then there is a morphism of species $\eta : \mathbf{x} \longrightarrow \mathbf{y}$ such that $\mathrm{el}(\eta) = \rho$: send an $\mathbf{x}$-structure s on a set I to $\rho(s)$, which is a $\mathbf{y}$-structure on the set I by the condition on ρ.

We write $\pi_0(\mathbf{x})$ for the set of connected components of the grupoid $\mathrm{el}(\mathbf{x})$ and call its elements *types of* $\mathbf{x}$*-structures.* If s is an $\mathbf{x}$-structure, we denote its type by $|s|$. One can (and should) think about the type of a structure as being obtained from it by deleting labels. To illustrate, the following two posets over $[2]$ are not equal, but have the same type, since the transposition $(12) \in \mathrm{Aut}([2])$ transports one to the other:



3. Examples

To understand all that follows it useful to have a list of examples in mind. We collect in this section such a list. We also include some examples of morphisms. For a comprehensive treatment of combinatorial species, we refer the reader to the book [**LBL1998**].

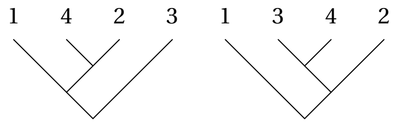
E1. The *exponential* or *uniform* species $\mathbf{e} : \mathsf{Set}^\times \longrightarrow \mathsf{FinSet}$ is the species that assigns to every finite set I the singleton set $\{I\}$, and to any bijection $\sigma : I \longrightarrow J$ the unique bijection $\mathbf{e}(\sigma) : \mathbf{e}(I) \longrightarrow \mathbf{e}(J)$. Remark that $\mathbf{e}$ is the unique species, up to isomorphism, that has exactly one structure over each finite set. For ease of notation, we will write $*_I$ for $\{I\}$.

E2. The *species of partitions* Π assigns to each finite set I the collection of partitions of I: sets $T = \{T_1, \ldots, T_s\}$ of nonempty disjoint subsets of I whose union is I. If $\sigma : I \longrightarrow J$ is a bijection and T is a partition of I, $\Pi(\sigma)(T) = \{\sigma T_1, \ldots, \sigma T_s\}$ is the partition of J obtained by transporting T along σ.

E3. The *species of compositions* Σ assigns to each finite set I the collection of composition of I: ordered tuples $(F_1, \ldots, F_t)$ of nonempty disjoint subsets of I whose union is I. If $\sigma : I \longrightarrow J$ is a bijection and F is a composition of I, $\Sigma(\sigma)(F) = (\sigma F_1, \ldots, \sigma F_t)$ is the composition of J obtained by transporting F along σ.

E4. The *species of derangements* Der assigns to each finite set I the collection of bijections $\sigma : I \longrightarrow I$ with no fixed points, and transports structures by conjugation: if $\sigma : I \longrightarrow I$ is a derrangement and $\tau : I \longrightarrow J$ is a bijection, then $\mathsf{Der}(\tau)(\sigma) = \tau\sigma\tau^{-1}$. There is also the *species of permutations* Per, that assigns to each set I the set $\mathsf{Per}(I)$ of bijections of I, and this is in turn a subspecies of the *species of endofunctions* End, that assigns to each set I the set $\mathsf{End}(I)$ of functions of I to itself.

E5. There is a species $\mathscr{A}$ that assigns to each finite set I the collection $\mathscr{A}(I)$ of binary trees with leaves bijectively labelled by I — alternatively, $\mathscr{A}(I)$ consists of properly parenthesised words with letters in I. To illustrate, the words $(1(42))3$ and $1((34)2)$ over $\{1,2,3,4\}$ correspond to the trees



If $\sigma : I \longrightarrow J$ is a bijection and $z \in \mathscr{A}(I)$ is a labelled binary tree, $\mathscr{A}(\sigma)(z)$ relabels the leaves of z according to σ.

E6. There is a species Simp that assigns to each set I the collection of simplicial structures on I, this is, collections of finite subsets $S \subseteq 2^I$ that contain all singleton sets of elements of I, and such that whenever $\Delta \in S$ and $\Delta' \subseteq \Delta$, then $\Delta' \in S$. We call the elements of S *simplices*.

E7. Let X be topological space. There is a species $\mathscr{C}_X$ that assigns to each finite set I the collection of continuous functions $X^I \longrightarrow X$. If $\sigma : I \longrightarrow J$ is a bijection and $f : X^I \longrightarrow X$ is a morphism we set $\sigma f((x_j)) = f((x_{\sigma^{-1}j}))$.

E8. Again, let X be a topological space. There is a species $\mathscr{F}_X$ that assigns to each finite set I the configuration space $\mathscr{F}_X(I) \subseteq X^I$ of X with coordinates on I: $\mathscr{F}_X(I)$ consists of tuples $(x_i)_{i\in I}$ with $x_i \neq x_j$ whenever i and j are distinct elements of I. As in the previous example, there is an obvious action of any bijection $\sigma : I \longrightarrow J$ that permutes the coordinates. For each fixed finite set I, the set of types of structures over I is usually called the *unordered configuration space* $\mathscr{E}_X(I)$.

E9. There is a *species of parts* $\wp$ that sends each finite set I to the collection 2^I of parts of I, and sends each bijection $\sigma : I \longrightarrow J$ to the induced bijection $\sigma_* : 2^I \longrightarrow 2^J$. In a similar way, if n is a positive integer, there is a species $\wp_n$ which sends each finite set I to the set $\wp_n$ of its subsets of cardinality n; notice that $\wp_n(I)$ is empty if I has less than n elements, and that $\wp_n$ is a subspecies of $\wp$ for each n.

E10. A graph with vertices on a set I is a pair (I,E) where E is a collection of 2-subsets of I. For each finite set I, let $\mathsf{Gr}(I)$ be the collection of graphs on I. If $\sigma : I \longrightarrow J$ is a bijection and (I,E) is a graph on I, we set $\mathsf{Gr}(\sigma)(I,E) = (J,\sigma(E))$. This defines the *species* Gr *of graphs*.

E11. For each finite set I, let $L(I)$ be the collection of linear orders on I. If $\sigma : I \longrightarrow J$ is a bijection, we let $L(\sigma)$ send a linear order $i_1 i_2 \cdots i_t$ on the set I to the linear order $\sigma(i_1)\cdots\sigma(i_t)$ on J. This defines the *species L of linear orders.*

E12. A tree is a connected graph with no cycles, and the *species* Tr *of trees* is a subspecies of the species of graphs. A *rooted tree* on a set I is a t tree with vertices on I and a distinguished vertex $r \in I$, called the root of t. The species of rooted trees Tr^* assigns to each finite set I the collection of rooted trees on I, and transport of structures is done by relabelling a tree t according to a given bijection $I \longrightarrow J$.

E13. A *contraction* on a set I is a function $f : I \to I$ for which there exists a fixed point $x_0 \in I$ such that, for every $i \in I$, there is n for which $f^n(i) = x_0$. There is defined a *species* Con *of contractions* that is a subspecies of the species of endofunctions, with transport of structure is carried out by conjugation.

The following illustrates how combinatorial constructions on structures are encoded by the morphisms of species.

E14. The species of rooted trees Tr^* is isomorphic to the species Con of contractions. The isomorphism is obtained by assigning to each rooted tree the endofunction on the set of its labels that maps an element to its 'immediate successor' in the direction of the root. The inverse assignment maps a contraction $f : I \longrightarrow I$ with distinguished point x_0 to the rooted tree with root x_0, in which there is an edge $i \to j$ whenever $f(i) = j$.

E15. The species of derrangements is a subspecies of the species of permutations, and this is in turn a subspecies of the species of endofunctions.

E16. The species of trees is both a subspecies of the species of graphs and of the species of rooted trees, and the species of graphs is a subspecies of the species of simplicial structures.

E17. If g is a graph on a finite set I, the collection of its connected components is a partition of I. This defines a morphism of species $\mathsf{Gr} \longrightarrow \Pi$.

E18. There is a morphism of species $L \longrightarrow \Sigma$ that assigns to a linear order $i_1 \cdots i_n$ on I the corresponding composition of I into singletons, and there is a morphism of species $\Sigma \longrightarrow \Pi$ that assigns to a composition F of a set I the partition on I obtained by forgetting the order of the blocks of F.

CHAPTER II

Monoidal categories and simplicial homology

1. Monoidal categories

Let C be a category and let $\otimes$ be a functor $\mathsf{C}\times\mathsf{C}\longrightarrow\mathsf{C}$. A natural isomorphism $a : (?\otimes ?)\otimes ? \longrightarrow ?\otimes(?\otimes ?)$ is called an *associator*. We say a satisfies the *pentagonal axiom* if the diagram

$$\begin{tikzcd}
(x\otimes(y\otimes z))\otimes w \arrow[dd, "a"] & ((x\otimes y)\otimes z)\otimes w \arrow[l, "a\otimes 1"'] \arrow[d, "a"] \\
 & (x\otimes y)\otimes(z\otimes w) \arrow[d, "a"] \\
x\otimes((y\otimes z)\otimes w) \arrow[r, "1\otimes a"] & x\otimes(y\otimes(z\otimes w))
\end{tikzcd}$$

commutes for every four objects w,x,y,z in C. A *unit object* for $\otimes$ is an object e of C with a choice of natural isomorphisms $r : ?\otimes e\longrightarrow ?$ and $l : e\otimes ?\longrightarrow ?$. We say (e,r,l) satisfies the *triangle axiom* with respect to $(\otimes,a)$ if the diagram

$$\begin{tikzcd}
(x\otimes e)\otimes y \arrow[rr, "a"] \arrow[dr, "r\otimes 1"'] & & x\otimes(e\otimes y) \arrow[dl, "1\otimes l"] \\
 & x\otimes y &
\end{tikzcd}$$

commutes for each pair of objects x,y in C. A monoidal category is the data of a 6-tuple $(\mathsf{C},\otimes,a,e,r,l)$ with C a category, $\otimes:\mathsf{C}\times\mathsf{C}\longrightarrow\mathsf{C}$ a bifunctor, a an associator for $\otimes$ and (e,r,l) a unit for $(\otimes,a)$, which satisfy the pentagonal and triangle axioms. We will usually denote a monoidal category by $(\mathsf{C},\otimes,e)$ without explicit mention of the associator or the isomorphisms r and l.

A monoidal category is *strict* if the isomorphisms a,r and l are identities. Our canonical examples for monoidal categories are the category Set with the cartesian product and unit object $e=\{\varnothing\}$, and the category of modules ${}_k\mathsf{Mod}$ over k with the usual tensor product $\otimes_k$ and unit object the base ring k. Any category with finite products is monoidal, with the product taking the role of the tensor product and the final object that of the unit object, and, dually, any category with finite coproducts is monoidal. Such categories are called *(co)cartesian monoidal categories*. The category of endofunctors of a category C is a strict monoidal category, with product the composition and unit object the identity functor.

Let $(\mathsf{C}, \otimes, e)$ be a monoidal category. A *comonoid* in C is an object c of C endowed with two arrows $\Delta : c \longrightarrow c \otimes c$ and $\varepsilon : c \longrightarrow e$ so that Δ is *coassociative* and ε is *counital with respect to* Δ in the sense that the diagrams

$$\begin{tikzcd} c \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & c \otimes c \arrow[d, "1\otimes\Delta"] \\ c \otimes c \arrow[r, "\Delta\otimes 1"'] & c \otimes c \otimes c \end{tikzcd} \qquad \begin{tikzcd} c \otimes e & & c \otimes c \arrow[d, "\varepsilon\otimes 1"] \\ & c \arrow[ul, "r^{-1}"'] \arrow[ur, "\Delta"] \arrow[dl, "\Delta"] \arrow[dr, "l^{-1}"'] & \\ c \otimes c \arrow[uu, "1\otimes\varepsilon"] & & e \otimes c \end{tikzcd}$$

are commutative. Remark that implicit in the writing of a triple product is the use of the arrows provided by the associator a. Dual diagrams provide the definition of a monoid object in C. In our canonical examples, a monoid in Set is a classical monoid, and a monoid in ${}_k\mathsf{Mod}$ is an associative and unital k-algebra. If k is a field and A is a finite dimensional k-algebra with multiplication $\mu : A \otimes A \longrightarrow A$ and unit $\eta : k \longrightarrow A$, the dual vector space $A' = \hom_k(A, k)$ is a comonoid in ${}_k\mathsf{Mod}$ —what is usually called a k-coalgebra— with counit $\varepsilon = \eta'$ and comultiplication given by the composition $A' \xrightarrow{\mu'} (A \otimes A)' \xrightarrow{\simeq} A' \otimes A'$. Every set X is a comonoid in a unique way, by means of the diagonal map $X \longrightarrow X \times X$ and the unique constant map $X \longrightarrow \{\varnothing\}$.

If (c, Δ, ε) and $(c', \Delta', \varepsilon')$ are comonoids in C, a *morphism of comonoids*[1] is an arrow $f : c \longrightarrow c'$ in C that renders the following two diagrams commutative:

$$\begin{tikzcd} c \arrow[r, "\Delta"] \arrow[d, "f"'] & c \otimes c \arrow[d, "f\otimes f"] \\ c' \arrow[r, "\Delta'"'] & c' \otimes c' \end{tikzcd} \qquad \begin{tikzcd} c \arrow[dr, "\varepsilon"] \arrow[dd, "f"'] & \\ & e \\ c' \arrow[ur, "\varepsilon'"'] & \end{tikzcd}$$

A left c-comodule x is an object of C endowed with an arrow $\lambda : x \longrightarrow c \otimes x$, called a *left coaction*, subject to the commutativity of the leftmost square in Figure 1. A right c-comodule with a right coaction $\rho : x \longrightarrow x \otimes c$ is defined analogously. If x is both a left and a right c-comodule, we shall say the two coactions are *compatible* if the second square in Figure 1 commutes, and the triple (x, λ, ρ) is then said to be a c-bicomodule.

[1] At this point, we hope the reader can guess what a morphism of monoids is supposed to be.

$$
\begin{tikzcd}
x \arrow[r, "\lambda"] \arrow[d, "\lambda"'] & c \otimes x \arrow[d, "\Delta\otimes 1"] \\
c \otimes x \arrow[r, "1\otimes\lambda"] & c \otimes c \otimes x
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[r, "\lambda"] \arrow[d, "\rho"'] & c \otimes x \arrow[d, "1\otimes\rho"] \\
x \otimes c \arrow[r, "\lambda\otimes 1"] & c \otimes x \otimes c
\end{tikzcd}
$$

$$
\begin{tikzcd}
x \arrow[r, "g"] \arrow[d, "\lambda"'] & x' \arrow[d, "\lambda'"] \\
c \otimes x \arrow[r, "1\otimes g"] & c \otimes x'
\end{tikzcd}
$$

FIGURE 1. The diagrams defining left coassociativity, compatibility of coactions and morphisms of left comodules.

If (x,λ) and (x',λ') are left c-comodules, a *morphism of left c-comodules* is an arrow $g : x \longrightarrow x'$ in C that renders the third square in Figure 1 commutative. In a similar fashion, we define morphisms of right c-comodules and of c-bicomodules, and in this way we obtain the categories ${}^c\mathrm{mod}$, mod^c and ${}^c\mathrm{mod}^c$ of left c-comodules, right c-comodules and c-bicomodules. Every comonoid c is a bicomodule over itself —in particular, a left and a right c-comodule— with the comultiplication map $\Delta : c \longrightarrow c \otimes c$ playing the role of both the left and the right coactions. Dually, every monoid a is a bimodule over itself, and there are categories ${}_a\mathrm{mod}$, mod_a and ${}_a\mathrm{mod}_a$ of left a-modules, right a-modules, and a-bimodules.

Let $\tau : \mathsf{C} \times \mathsf{C} \longrightarrow \mathsf{C} \times \mathsf{C}$ be the flip functor that sends a pair (x,y) to (y,x) and acts in the obvious way on morphisms. We say that C is *braided* if it is endowed with a *braiding*, that is, a natural isomorphism $b : (?\otimes ?) \longrightarrow (?\otimes ?)\circ\tau$ that satisfies the following *hexagonal axiom*: for any choice of objects x, y, z in C, the diagrams

$$
\begin{tikzcd}
& x \otimes (y \otimes z) \arrow[r, "b"] & (y \otimes z) \otimes x \arrow[dr, "a"] & \\
(x \otimes y) \otimes z \arrow[ur, "a"] \arrow[dr, "b\otimes 1"'] & & & y \otimes (z \otimes x) \\
& (y \otimes x) \otimes z \arrow[r, "a"] & y \otimes (x \otimes z) \arrow[ur, "1\otimes b"'] &
\end{tikzcd}
$$

$$\begin{tikzcd}
 & (x\otimes y)\otimes z \arrow[r, "b"] & z\otimes(x\otimes y) \arrow[dr, "a^{-1}"] & \\
x\otimes(y\otimes z) \arrow[ur, "a^{-1}"] \arrow[dr, "1\otimes b"'] & & & (z\otimes x)\otimes y \\
 & x\otimes(z\otimes y) \arrow[r, "a^{-1}"] & (x\otimes z)\otimes y \arrow[ur, "b\otimes 1"'] &
\end{tikzcd}$$

commute. If additionally $b(y,x)\circ b(x,y) = \mathrm{id}_{x\otimes y}$ for any pair of objects x, y in C, we say that C is *symmetric*. A monoid (m,μ,η) in C is *commutative* if $\mu b = \mu$, and dually, a comonoid (c,Δ,ε) in C is *cocommutative* if $\Delta = b\Delta$. If m is a commutative monoid, any left module (x,λ) over m admits a bimodule structure with right action $\rho = \lambda b$. Dually, a left comodule over a cocommutative comonoid is canonically a bicomodule.

The categories of monoids and comonoids over C are monoidal: for example, if (c,Δ,ε) and $(c',\Delta',\varepsilon')$ are comonoids, then $c\otimes c'$ is a comonoid with comultiplication the composition $\Delta'' = 1\otimes b\otimes 1\circ\Delta\otimes\Delta'$ and counit $r\circ\varepsilon\otimes\varepsilon'$, the first structure map illustrated by the following diagram:

$$\begin{tikzcd}
c\otimes c' \arrow[dd, dashed, "\Delta''"] \arrow[dr, "\Delta\otimes\Delta'"] & \\
 & c\otimes c\otimes c'\otimes c' \arrow[dl, "1\otimes b\otimes 1"] \\
c\otimes c'\otimes c\otimes c' &
\end{tikzcd}$$

A *bimonoid* in C is a monoid in the category of comonoids in C or, equivalently, a comonoid in the category of monoids in C. Explicitly, this is an object z endowed both with the structure of a monoid (z,μ,η) and of a comonoid (z,Δ,ε) in such a way that μ and η are morphisms of comonoidsm, or equivalently, in such a way that Δ and ε are morphisms of monoids. If c is a comonoid and m is a monoid, the set $\hom_{\mathsf{C}}(c,m)$ is a classical monoid with product the operation $?\star?$ of *convolution* such that for $f,g\in\hom_{\mathsf{C}}(c,m)$, $f\star g : c\longrightarrow m$ is the composition

$$c \xrightarrow{\Delta} c\otimes c \xrightarrow{f\otimes g} m\otimes m \xrightarrow{\mu} m$$

and with unit the composition $\eta\varepsilon : c\longrightarrow e\longrightarrow m$. In particular, the set of endomorphisms of a bimonoid z is in this way a classical monoid with operation this convolution, and we say z is a *Hopf monoid* if the identity id_z is invertible there. In this case, the inverse of id_z is called the *antipode* of z and is usually denoted by s. The simplest example of a Hopf monoid is the group algebra kG of a group G over k, which is a Hopf monoid in ${}_k\mathsf{Mod}$, with comultiplication $\Delta(g) = g \otimes g$, counit $\varepsilon : kG \longrightarrow k$ such that $\varepsilon(g) = 1$ for $g \in G$, and antipode $s : kG \longrightarrow kG$ such that $s(g) = g^{-1}$ for $g \in G$. Similarly, the polynomial algebra $k[x]$ is a Hopf monoid in the same category, now with $\Delta(x) = x \otimes 1 + 1 \otimes x$, $\varepsilon(x) = 0$ and $s(x) = -x$. For more exotic examples of Hopf algebras and an extensive treatment of monoidal categories, we refer the reader to [Kas1995].

If z is a bimonoid, every object x in C may be endowed with the left and right z-comodule structures given by the compositions

$$x \xrightarrow{l^{-1}} e \otimes x \xrightarrow{\eta \otimes 1} z \otimes x \qquad\qquad x \xrightarrow{r^{-1}} x \otimes e \xrightarrow{1 \otimes \eta} x \otimes z$$

and with the left and right z-module structures

$$z \otimes x \xrightarrow{\varepsilon \otimes 1} e \otimes x \xrightarrow{l} x \qquad\qquad x \otimes z \xrightarrow{1 \otimes \varepsilon} x \otimes e \xrightarrow{r} x.$$

We say these are the *trivial* (co)actions on x, and say that x is a *trivial* (co)module. These actions are compatible and turn x into a *trivial* z-bimodule and a *trivial* z-bicomodule. In a similar fashion, every left z-module can be made into a z-bimodule with a right trivial action, and every left z-comodule can be made into a z-bicomodule with a right trivial action.

If z is a bimonoid in C, each of the six possible categories of (bi)(co)modules over z is itself monoidal with tensor product induced by that of C: to illustrate, if x and x' are left z-comodules, then the tensor product $x \otimes x'$ is a left z-comodule with left coaction λ'' the unique arrow which makes the following diagram commute

$$\begin{tikzcd}
x \otimes x' \arrow[r, "\lambda \otimes \lambda'"] \arrow[d, dashed, "\lambda''"] & z \otimes x \otimes z \otimes x' \arrow[d, "1 \otimes b \otimes 1"] \\
z \otimes x \otimes x' & z \otimes z \otimes x \otimes x' \arrow[l, "\mu \otimes 1"]
\end{tikzcd}$$

2. Linear and abelian categories

A category C is a *preadditive category* or an ***Ab**-category* if, for every pair of objects x and y in C, the set $\hom(x, y)$ is an abelian group, and the law of composition of morphisms is compatible with this structure, meaning that for every three objects x, y and z in C, the composition of morphisms

$$?\circ? : \hom(y,z)\times\hom(x,y)\longrightarrow\hom(x,z)$$

is a biadditive map. More generally, a *k-linear category* is a category whose hom-sets are k-modules and whose law of composition is k-bilinear; thus a $\mathbb{Z}$-linear category is exactly an **Ab**-category.

An *additive category* is a preadditive category C that has a zero object, and such that every two objects x and y in C admit a biproduct. The opposite category of a k-linear category is canonically a k-linear category, and since zero objects and biproducts are self-dual concepts, the opposite of an additive category is itself additive. An additive category is *abelian* if

AB1. every morphism has a kernel and a cokernel,
AB2. every monic morphism is the kernel of its cokernel, and
AB3. every epic morphism is the cokernel of its kernel.

These axioms are self dual, and therefore the opposite category of an abelian category is itself abelian. The following conditions may or may not be satisfied by an abelian category. We state them for future reference:

AB4. Every family of objects admits a coproduct.
AB5. The coproduct of a family of monomorphisms is a monomorphism.
AB6. The filtered colimit of a family of exact sequences is exact.

We will use a star to denote the dual of one of the axioms above. For example, **AB2*** is **AB3**, and **AB5*** states the product of a family of epimorphisms is an epimorphism.

PROPOSITION 2.1. *If A is an abelian category and C is a locally small category, the category of functors $\mathsf{C}\longrightarrow\mathsf{A}$ is abelian, and it satisfies **AB4** (or, respectively, **AB4***) if A does.*

In particular, Sp_k is an abelian k-linear category and satisfies axioms **AB4*** and **AB4**. More generally, the category $\mathsf{Sp}(\mathsf{C})$ of species over C usually inherits whatever extra structure C has. The following proposition lists two instances of this which are of use to us:

PROPOSITION 2.2. *Let C be a category.*

SP1. *If* C *is* k*-linear, the category* $\mathsf{Sp}(\mathsf{C})$ *of species over* C *has a canonical structure of* k*-linear category induced from that of* C*. Morover, if* C *is additive or abelian, then so is* $\mathsf{Sp}(\mathsf{C})$*.*

SP2. *If* C *is monoidal with tensor product* $\otimes$*, there is an induced monoidal structure on* $\mathsf{Sp}(\mathsf{C})$ *with tensor product, denoted by* $\times$*, which on objects is as follows: if* $\mathbf{x}$ *and* $\mathbf{y}$ *are species over* C*, then* $(\mathbf{x}\times\mathbf{y})(I)=\mathbf{x}(I)\otimes\mathbf{y}(I)$ *for all finite sets* I*. If* C *is braided monoidal, so is* $\mathsf{Sp}(\mathsf{C})$*, and if additionally it is symmetric, the same is true for* $\mathsf{Sp}(\mathsf{C})$*.*

We call the product $\times$ the *Hadamard product* or *pointwise product* of $\mathsf{Sp}(\mathsf{C})$. It is worth remarking that under the equivalence K in (1), the Hadamard product of two species $\mathbf{x}$ and $\mathbf{y}$ satisfies

$$K(\mathbf{x}\times\mathbf{y})([n]) = \operatorname{Res}_{S_n}^{S_n\times S_n}(\mathbf{x}([n])\times\mathbf{y}([n])),$$

where we view S_n as a subgroup of $S_n\times S_n$ via the diagonal embedding.

PROOF. The proof is standard and we omit it. ◀

$(\mathsf{C},\otimes,e,a,r,l,b)$ so that C is both a braided monoidal cateory and an abelian category, in such a way that $\otimes$ is biadditive.

3. The Cauchy monoidal structure on species

A *decomposition* S of *length* q of a set I is an ordered tuple $(S_1,\dots,S_q)$ of possible empty subsets of I, which we call the *blocks of* S, that are pairwise disjoint and whose union is I. We say S is a *composition* of I if every block of S is nonempty. It is clear that if I has n elements, every composition of I has at most n blocks. We will write $S\vdash I$ to mean that S is a decomposition of I, and if necessary will write $S\vdash_q I$ to specify that the length of S is q. Notice the empty set has exactly one composition which has length zero, the empty composition, and exactly one decomposition of each length $n\in\mathbb{N}_0$. If T is a subset of I and $\sigma: I\longrightarrow J$ is a bijection, we let $\sigma_T: T\longrightarrow\sigma(T)$ be the bijection induced by σ.

Proposition 2.2 tells us that the category Sp_k of species over ${}_k\mathrm{Mod}$ is abelian and monoidal with respect to the Hadamard product. We will construct another product in Sp_k, called the *Cauchy product* and denoted by $\otimes$, which will play a central role in all that follows, and which categorifies the usual Cauchy product of power series.

Let $\mathbf{x}$ and $\mathbf{y}$ be linear species over k. The Cauchy product $\mathbf{x}\otimes\mathbf{y}$ is the linear species such that for every finite set I

$$(\mathbf{x}\otimes\mathbf{y})(I) = \bigoplus_{(S,T)\vdash I} \mathbf{x}(S)\otimes\mathbf{y}(T),$$

the direct sum running through all decompositions of I of length two, and for every bijection $\sigma : I \longrightarrow J$

$$(\mathbf{x}\otimes\mathbf{y})(\sigma) = \bigoplus_{(S,T)\vdash I} \mathbf{x}(\sigma_S)\otimes\mathbf{y}(\sigma_T).$$

As it happens with the Hadamard product, the Cauchy product is better understood when viewing species as the product of representations of the various symmetric groups. Indeed, for each n and each pair (p,q) with $p+q=n$, there is an isomorphism

$$\bigoplus_{S\subseteq I,\#S=p} \mathbf{x}(S)\otimes\mathbf{y}(T) \simeq \operatorname{Ind}_{S_p\times S_q}^{S_{p+q}}(\mathbf{x}([p])\otimes\mathbf{y}([q])),$$

and these collect to give an isomorphism

$$(\mathbf{x}\otimes\mathbf{y})([n]) \simeq \bigoplus_{p+q=n} \operatorname{Ind}_{S_p\times S_q}^{S_{p+q}}(\mathbf{x}([p])\otimes\mathbf{y}([q])).$$

It should be clear this construction extends to produce a bifunctor $\otimes : \mathsf{Sp}_k\times\mathsf{Sp}_k \longrightarrow \mathsf{Sp}_k$. In what follows, whenever we speak of the category Sp_k, we will view it as a monoidal category with the monoidal structure described in the following proposition.

PROPOSITION 3.1. *The Cauchy product* $\otimes$ *makes* Sp_k *into an* abelian braided monoidal category, *in the sense that it is abelian, braided monoidal, and* $\otimes$ *is* k*-bilinear. The unit object* $\mathbf{1}$ *is the linear species with* $\mathbf{1}(\varnothing)$ *the free* k*-module with basis* $\{\varnothing\}$ *and* $\mathbf{1}(I)=0$ *whenever* I *is nonempty. The associator, left and right unitors and braiding are all induced by those of* ${}_k\mathrm{Mod}$.

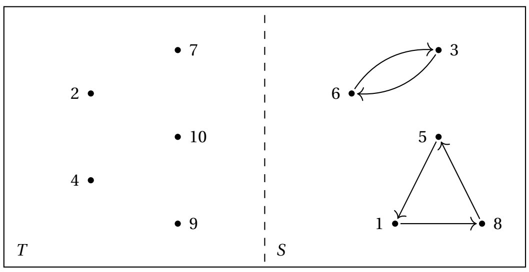
The series of verifications needed to prove this last proposition are tedious but otherwise standard. For convenience, we give a broad idea of how to endow Sp_k with such structure. We identify once and for all the vector space with basis the set $\{\varnothing\}$ with our ground ring k, by means of the isomorphism that sends the basis element $\varnothing$ to $1\in k$, so that $\mathbf{1}$ can be seen as the species concentrated on degree 0, where it has the value k. This has the effect of turning the unitors $r:\mathbf{x}\otimes\mathbf{1}\longrightarrow\mathbf{x}$ and $l:\mathbf{1}\otimes\mathbf{x}\longrightarrow\mathbf{x}$

into the usual isomorphism $k \otimes V \simeq V \otimes k \simeq V$ in each cardinal. In a similar fashion, the associator $a : (\mathbf{x} \otimes \mathbf{y}) \otimes \mathbf{z} \longrightarrow \mathbf{x} \otimes (\mathbf{y} \otimes \mathbf{z})$ is, for each choice of linear species $\mathbf{x}, \mathbf{y}$ and $\mathbf{z}$, obtained by a cardinalwise use of the usual natural isomorphism $a : (U \otimes V) \otimes W \longrightarrow U \otimes (V \otimes W)$ for k-modules U, V, W. The data $(\otimes, \mathbf{1}, a, r, l)$ makes Sp_k into a monoidal category. functor between such categories. Unless otherwise stated, we will reserve the symbol $\otimes$ for the Cauchy product of species. The usual braiding in ${}_k\mathsf{Mod}$ gives a braiding in Sp_k obtained by applying the braiding componentwise, and with this structure $(\mathsf{Sp}_k, \otimes, \mathbf{1}, a, r, l, b)$ is a braided monoidal category.

It is important to notice the construction of the Cauchy product in Sp_k carries over to the category $\mathsf{Sp}(\mathsf{C})$ when C is any monoidal category with finite coproducts which commute with its tensor product. The main example of this phenomenon happens when C is the category Set. If $\mathbf{x}$ and Y are set species, the species $\mathbf{x} \otimes \mathbf{y}$ has

$$(\mathbf{x} \otimes \mathbf{y})(I) = \bigsqcup_{(S,T)\vdash I} \mathbf{x}(S) \times \mathbf{y}(T),$$

so that a structure z of species $\mathbf{x} \otimes \mathbf{y}$ over a set I is determined by a decomposition (S, T) of I and a pair of structures (z_1, z_2) of species $\mathbf{x}$ and $\mathbf{y}$ over S and T, respectively. For example, every permutation τ of a finite set I determines a derangement on the subset S of points of I not fixed by τ and the set T of fixed points of τ, and (S, T) is a decomposition of I: this observation leads to the construction of an isomorphism $\mathsf{Per} \longrightarrow \mathsf{Der} \otimes E$ from the species of permutations Per to the product of the exponential species E and the species Der of derangements; we encourage the reader to exhibit this isomorphism explicitly, which we illustrate in the following figure for the permutation $(36)(158) \in S_{10}$:



The linearization functor $\mathscr{L} : \mathsf{Sp} \longrightarrow \mathsf{Sp}_k$ preserves the monoidal structures we have defined on these categories, in the sense there is a natural isomorphism $\mathscr{L}(\mathbf{x} \otimes \mathbf{y}) \longrightarrow \mathscr{L}\mathbf{x} \otimes \mathscr{L}\mathbf{y}$ for each pair of objects $\mathbf{x}, \mathbf{y}$ in Sp. For details on such *monoidal functors* see [**Kas1995**, XI. §4].

The following will be useful, and we record it for future reference: if $\mathbf{x}, \mathbf{y}_1, \ldots, \mathbf{y}_r$ are linear species, a map of species $\alpha : \mathbf{x} \longrightarrow \mathbf{y}_1 \otimes \cdots \otimes \mathbf{y}_r$ determines and is determined by a choice of equivariant k-module maps

$$\alpha_I : \mathbf{x}(I) \longrightarrow \bigoplus \mathbf{y}_1(S_1) \otimes \cdots \otimes \mathbf{y}_r(S_r),$$

one for each finite set I, the direct sum running through decompositions $(S_1, \ldots, S_r)$ of length r of I. In turn, the map α_I is specified uniquely by its *components* at each decomposition $S = (S_1, \ldots, S_r)$, which we denote $\alpha(S_1, \ldots, S_r)$ without further mention to the set I which is implicit, for $\bigcup S$ equals I. Moreover, it suffices to specify α_I for I the sets $[\![n]\!]$ with $n \in \mathbb{N}_0$. This said, we will usually define a map $\alpha : \mathbf{x} \longrightarrow \mathbf{y}_1 \otimes \cdots \otimes \mathbf{y}_r$ by specifying its components at each decomposition of length r of I.

4. Monoids, comonoids and bimonoids in species

A monoid $(\mathbf{x}, \mu, \eta)$ in the category Sp_k is determined by a multiplication $\mu : \mathbf{x} \otimes \mathbf{x} \longrightarrow \mathbf{x}$ and a unit $\eta : \mathbf{1} \longrightarrow \mathbf{x}$. Specifying the first amounts to giving its components $\mu(S, T) : \mathbf{x}(S) \otimes \mathbf{x}(T) \longrightarrow \mathbf{x}(I)$ at each decomposition (S, T) of every finite set I, and specifying the latter amounts to a choice of the element $\eta(\varnothing)(1) \in \mathbf{x}(\varnothing)$, which we will denote by 1 if no confusion should arise.

We think of the multiplication as an operation that glues partial structures on I, and of the unit as an "empty" structure. For example, the species of graphs admits a multiplication $k\mathsf{Gr} \otimes k\mathsf{Gr} \longrightarrow k\mathsf{Gr}$ which is the linear extension of the map that takes a pair of graphs $(g_1, g_2) \in \mathsf{Gr}(S) \times \mathsf{Gr}(T)$ and constructs the disjoint union $g_1 \sqcup g_2$ on I. The unit for this multiplication is the empty graph $\varnothing \in \mathsf{Gr}(\varnothing)$. One can readily check μ is associative and unital with respect to η, so we indeed have a monoid $k\mathsf{Gr}$.

A *linearized monoid* in Sp_k is a linearized species $\mathbf{x}$ of the form $k\mathbf{x}_0$ with $\mathbf{x}_0$ a monoid in the category Sp. For example, the monoid structure on the linearization of the species of graphs just defined is linearized. Simply put, a monoid is linearized when it is linearized as a species and its operations are also linearized. A morphism of *linearized monoids* is one obtained by linearization of a morphism of monoids in Sp.

Dually, a comonoid $(\mathbf{x}, \Delta, \varepsilon)$ in Sp_k is determined by a comultiplication $\Delta : \mathbf{x} \longrightarrow \mathbf{x} \otimes \mathbf{x}$ and a counit $\varepsilon : \mathbf{x} \longrightarrow \mathbf{1}$. The comultiplication has, at each decomposition (S, T) of I, a component $\Delta(S, T) : \mathbf{x}(I) \longrightarrow \mathbf{x}(S) \otimes \mathbf{x}(T)$, which we think of as breaking up a combinatorial structure on I into substructures on S and T, while the counit is a map of k-modules $\mathbf{x}(\varnothing) \longrightarrow k$.

To continue with our example, the linearization of the species of graphs admits a comultiplication $k\mathsf{Gr} \longrightarrow k\mathsf{Gr} \otimes k\mathsf{Gr}$ that sends a graph g on a set I to $g_S \otimes g_T \in k\mathsf{Gr}(S) \otimes k\mathsf{Gr}(T)$, where g_S and g_T are the subgraphs induced by g on S and T, respectively. This comultiplication admits as counit the morphism $\varepsilon : k\mathsf{Gr} \longrightarrow \mathbf{1}$ that assigns $1 \in k$ to the empty graph. In this way, we obtain a comonoid structure on $k\mathsf{Gr}$ which is, in fact, compatible with the monoid structure we described in the previous paragraph: we therefore have a bimonoid structure on $k\mathsf{Gr}$.

In general, defining a comonoid structure on a linear species $\mathbf{x}$ requires we give a map $\varepsilon : \mathbf{x}(\varnothing) \longrightarrow \mathbf{1}(\varnothing)$ and, for each finite set I and each decomposition (S,T) of I, a map $\Delta(S,T) : \mathbf{x}(I) \longrightarrow \mathbf{x}(S) \otimes \mathbf{x}(T)$, in such a way the following diagrams commute[2]

$$\begin{tikzcd}
\mathbf{x}(A) \otimes \mathbf{x}(B) \otimes \mathbf{x}(C) & \mathbf{x}(AB) \otimes \mathbf{x}(C) \arrow[l, "\Delta\otimes 1"'] \\
\mathbf{x}(A) \otimes \mathbf{x}(BC) \arrow[u, "1\otimes\Delta"] & \mathbf{x}(ABC) \arrow[l, "\Delta"] \arrow[u, "\Delta"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{1}(\varnothing) \otimes \mathbf{x}(I) & & \mathbf{x}(I) \otimes \mathbf{x}(\varnothing) \arrow[dd, "1\otimes\varepsilon"] \\
& \mathbf{x}(I) \arrow[ul] \arrow[ur, "\Delta"'] \arrow[dl, "\Delta"] \arrow[dr] & \\
\mathbf{x}(\varnothing) \otimes \mathbf{x}(I) \arrow[uu, "\varepsilon\otimes 1"] & & \mathbf{x}(I) \otimes \mathbf{1}(\varnothing)
\end{tikzcd}$$

This is a consequence of the fact that, in order to check the equality $1 \otimes \Delta \circ \Delta = \Delta \otimes 1 \circ \Delta$, it is sufficent that we do so at each decomposition (A,B,C) of length 3 of each finite set I, and that the counit $\varepsilon : \mathbf{x} \longrightarrow \mathbf{1}$ necessarily has $\varepsilon(I)$ the zero map when I is nonempty since $\mathbf{1}(I) = 0$ in such case, so the only data provided by ε is the map $\mathbf{x}(\varnothing) \longrightarrow \mathbf{1}(\varnothing) \simeq k$.

We can partially reverse this idea. A *pre-comonoid* in Sp is a set species $\mathbf{x}_0$ endowed with maps $\Delta(S,T) : \mathbf{x}_0(S \cup T) \longrightarrow \mathbf{x}_0(S) \times \mathbf{x}_0(T)$ for each decomposition (S,T) of each

[2] We are writing AB for a disjoint union $A \sqcup B$ to lighten up the notation.

finite set I, that render the diagrams obtained from the above two by replacing $\otimes$ with $\times$ commutative, where $\mathbf{x}_0(\varnothing) \longrightarrow \mathbf{1}(\varnothing)$ is the unique constant map. The point of this definition is that, upon linearization, the linear species $\mathbf{x} = k\mathbf{x}_0$ becomes a *bona fide* comonoid in Sp_k. Observe that the data defining a pre-comonoid in Sp does *not* define a comonoid in Sp —in fact, a set species $\mathbf{x}$ that is a comonoid in Sp is necessarily concentrated in the empty set: the existence of a counit determines maps $\varepsilon_I : \mathbf{x}(I) \longrightarrow \mathbf{1}(I)$, and $\mathbf{1}(I)$ is empty whenever I is not. A *linearized comonoid* will be, for us, the result of linearizing a pre-comonoid in Sp. It should be clear what a *morphism of pre-comonoids* is supposed to be, and a *morphism of linearized comonoids* is just the linearization of such a thing. If $\mathbf{x} = k\mathbf{x}_0$ is a linearized comonoid with underlying comultiplication $\Delta : \mathbf{x}_0 \longrightarrow \mathbf{x}_0 \otimes \mathbf{x}_0$, we will write $\Delta(S,T)(z) = (z \mathbin{\backslash\!\backslash} S, z \mathbin{/\!/} T)$.

We emphasize the fact that in the dual situation of monoids in Sp_k the commutativity of the diagrams dual to (**??**) and (**??**) is a strictly stronger condition than the usual associativity and unit conditions. Combining the notions of linearized monoids and linearized comonoids one obtains, as expected, the notion of *linearized bimonoids* and their morphisms. Our main example of a bimonoid in Sp_k is the provided by the following proposition.

PROPOSITION 4.1. *The linearized exponential species* $\mathbf{e}$ *is a linearized bimonoid with multiplication and comultiplication with components*

$$\mu(S,T) : \mathbf{e}(S) \otimes \mathbf{e}(T) \longrightarrow \mathbf{e}(I), \qquad \Delta(S,T) : \mathbf{e}(I) \longrightarrow \mathbf{e}(S) \otimes \mathbf{e}(T)$$

at each decomposition (S,T) *of a finite set* I *such that*

$$\mu(S,T)(*_S \otimes *_T) = *_I, \qquad \Delta(S,T)(*_I) = *_S \otimes *_T$$

and with unit and counit the morphisms $\varepsilon : \mathbf{e} \longrightarrow \mathbf{1}$ *and* $\eta : \mathbf{1} \longrightarrow \mathbf{e}$ *such that* $\varepsilon(*_\varnothing) = 1$ *and* $\eta(1) = *_\varnothing$.

PROOF. The verifications needed to prove this follow immediately from the fact that $\mathbf{e}(I)$ is a singleton for every finite set I. ◀

The exponential species plays a central role in the category of linearized bimonoids, as evinced by the following proposition.

PROPOSITION 4.2.

(1) *The exponential species* $\mathbf{e}$ *admits a unique structure of linearized bimonoid.*

(2) *If* $\mathbf{x}$ *is a pre-comonoid in* Sp*, the linearization of the unique morphism of species* $\mathbf{x} \longrightarrow \mathbf{e}$ *is a morphism of linearized comonoids.*

(3) *In particular, every linearized comonoid is canonically an* $\mathbf{e}$*-bicomodule.*

PROOF. If s is a singleton set and x is any set, there is a unique function $x \longrightarrow s$, and it follows from this, first, that the bimonoids structure defined on $\mathbf{e}$ is the only linearized bimonoid structure, and, second, that if $\mathbf{x}$ is a species in Sp, there is a unique morphism of species $\mathbf{x} \longrightarrow \mathbf{e}$. If $\mathbf{x}$ is a pre-comonoid in Sp, the following square commutes because $\mathbf{e}(S) \times \mathbf{e}(T)$ has one element:

$$\begin{array}{ccc} \mathbf{e}(I) & \xrightarrow{\Delta} & \mathbf{e}(S) \times \mathbf{e}(T) \\ \uparrow & & \uparrow \\ \mathbf{x}(I) & \xrightarrow{\Delta} & \mathbf{x}(S) \times \mathbf{x}(T), \end{array}$$

and, by the same reason, $\mathbf{x} \longrightarrow \mathbf{e}$ is pre-counital. All this shows that the exponential species $\mathbf{e}$ is terminal in the category of linearized comonoids. This completes the proof of the proposition. ◀

We will fix some useful notation to deal with comonoids. Let $\mathbf{x} = k\mathbf{x}_0$ be a linearized species that is a comonoid in Sp_k; notice that we do not require it be a linearized comonoid. If z is an element of $\mathbf{x}_0(I)$, we write the image $\Delta(I)(z)$ as a sum $\sum z \backslash\!\backslash S \otimes z /\!/ T$ with $z \backslash\!\backslash S \otimes z /\!/ T$ denoting an element of $\mathbf{x}(S) \otimes \mathbf{x}(T)$ (not necessarily an elementary tensor, à la Sweedler).

Consider now a left $\mathbf{e}$-comodule $\mathbf{x}$ with coaction $\lambda : \mathbf{x} \longrightarrow \mathbf{e} \otimes \mathbf{x}$. Since $\mathbf{e}(S) = k\{*_S\}$, the component $\mathbf{x}(I) \longrightarrow \mathbf{e}(S) \otimes \mathbf{x}(T)$ can canonically be viewed as map $\mathbf{x}(I) \longrightarrow \mathbf{x}(T)$ which we denote by λ_T^I, and call the it *the restriction from* I *to* T *to the right.*

In these terms, that λ be counital means λ_I^I is the identity for all finite sets I, and the equality $1 \otimes \lambda \circ \lambda = \Delta \otimes 1 \circ \lambda$, which expresses the coassociativity of λ, translates to the condition that we have $\lambda_A^I = \lambda_A^B \circ \lambda_B^I$ for any chain of finite sets $A \subseteq B \subseteq I$. It follows that, if $\mathsf{FinSet}^{\mathrm{inc}}$ is the category of finite sets and inclusions, a left $\mathbf{e}$-comodule $\mathbf{x}$ in Sp_k can be viewed as a pre-sheaf $\mathsf{FinSet}^{\mathrm{inc}} \longrightarrow {}_k\mathrm{Mod}$. When convenient, we will write $z /\!/ S$ for $\lambda_S^I(z)$ without explicit mention to I, which will usually be understood from context. Using this notation, we can write the coaction on $\mathbf{x}$ as

$$\lambda(I)(z) = \sum e_S \otimes z /\!/ T.$$

Of course the same consideration apply to a right $\mathbf{e}$-comodule, and we write $z \mathbin{\backslash\!\backslash} T$ for $\rho^I_T(z)$. If $\mathbf{x}$ is both a left and a right $\mathbf{e}$-comodule with coactions λ and ρ, the compatibility condition for it to be an $\mathbf{e}$-bicomodule is that, for any finite set I and pair of non-necessarily disjoint subsets S, T of I, we have $\rho^S_{S\cap T}\lambda^I_S = \lambda^T_{S\cap T}\rho^I_T$. Schematically, we can picture this as follows:

$$\begin{tikzcd}
 & S\cap T & \\
S \arrow[ur, "\rho"] & & T \arrow[ul, "\lambda"'] \\
 & I \arrow[ul, "\lambda"] \arrow[ur, "\rho"'] &
\end{tikzcd}$$

There is a category $\mathsf{FinSet}^{\mathrm{binc}}$ such that an $\mathbf{e}$-bicomodule is exactly the same as a presheaf $\mathsf{FinSet}^{\mathrm{binc}} \longrightarrow \mathsf{Sp}_k$; we leave its construction to the categorically inclined reader. If the structure on $\mathbf{x}$ is cosymmetric, we will write $z \parallel S$ for the common value of $z \mathbin{\backslash\!\backslash} S$ and $z \mathbin{/\!/} S$. There is a close relation between linearized comonoids and linearized $\mathbf{e}$-bicomodules, as described in the following proposition.

PROPOSITION 4.3. *Let* $(\mathbf{x}, \Delta)$ *be a linearized comonoid, and let* $f_{\mathbf{x}} : \mathbf{x} \longrightarrow \mathbf{e}$ *be the unique morphism of linearized comonoids described in Proposition 4.2. There is on* $\mathbf{x}$ *an* $\mathbf{e}$*-bicomodule structure so that the coactions* $\lambda : \mathbf{x} \longrightarrow \mathbf{e}\otimes\mathbf{x}$ *and* $\rho : \mathbf{x} \longrightarrow \mathbf{x}\otimes\mathbf{e}$ *are obtained from postcomposition of* Δ *with* $f_{\mathbf{x}} \otimes 1$ *and* $1 \otimes f_{\mathbf{x}}$*, respectively.* ◀

We refer the reader to [**AM2010**, Chapter 8, §3, Proposition 29]. Remark that, with this proposition at hand, the notation introduced for bicomodules and that introduced for comonoids is consistent.

5. Hopf monoids in the category of species

Let $\mathbf{x}$ be a bimonoid in Sp_k with structure maps Δ and μ. We have already described in Section 1 how $\mathrm{End}(\mathbf{x})$ is a monoid under the convolution operation constructed from Δ and μ, and unit the composition $\eta\varepsilon$. It is readily seen convolution is biadditive with respect to the k-linear structure in $\mathrm{End}(\mathbf{x})$, so in fact $\mathrm{End}(\mathbf{x})$ is a k-algebra. Recall that a species $\mathbf{x}$ is *connected* if $\mathbf{x}(\varnothing)$ is free of rank one. The following result in [**AM2010**] states every connected bimonoid in Sp_k is automatically a Hopf monoid, and this automatically endows the various categories of representations of $\mathbf{x}$ with extra structure, as described in Section 1. More generally, a bimonoid $\mathbf{x}$ in Sp_k is a Hopf monoid precisely when $\mathbf{x}(\varnothing)$ is a Hopf k-algebra, and the antipode of $\mathbf{x}$ can, in that case, be explicitly constructed from the antipode of $\mathbf{x}(\varnothing)$ —this is a variant of what is known as *Takeuchi's theorem*, see [**AM2013**, Proposition 9].

THEOREM II.5.1. *Let* $(\mathbf{x}, \mu, \Delta)$ *be a bimonoid in* Sp_k.

(1) If $\mathbf{x}$ *is a Hopf monoid with antipode* s*, then* $\mathbf{x}(\varnothing)$ *is an Hopf* k*-algebra with antipode* $s(\varnothing)$.

(2) If $\mathbf{x}(\varnothing)$ *is a Hopf* k*-algebra with antipode* s_0*, then* $\mathbf{x}$ *is a Hopf monoid, and* s *can be iteratively constructed from* s_0, μ *and* Δ.

(3) In particular, if $\mathbf{x}$ *is a connected bimonoid,* $\mathbf{x}$ *is a Hopf monoid.*

PROOF. For a proof and an explicit formula for s in terms of s_0, we refer the reader to [**AM2010**, Chapter 8, §3.2, Proposition 8.10, and §4.2, Proposition 8.13]. The third part follows from the second since k is, in a unique way, a Hopf k-algebra. ◀

We define some connected bimonoids that will be of interest in Chapter III. In view of the previous result, they are all Hopf monoids in the category of linear species. Remark that, since the monoidal category Sp_k is symmetric, the tensor product of Hopf monoids in Sp_k is again Hopf monoid, so the following examples provide further ones by combining them into products. For completeness, we also list the bimonoid structure of $\mathbf{e}$. In all cases the unit and counit are the projection and the inclusion of $\mathbf{1}$ in the component of $\varnothing$.

H1. The exponential species $\mathbf{e}$ admits a bimonoid structure such that

- multiplication is given by the union of sets: $\mu(S,T)(e_S, e_T) = e_{ST}$
- comultiplication is given by partitioning a set into disjoint subsets: for $I = S \sqcup T$ a finite set $\Delta(S,T)(e_I) = e_S \otimes e_T$

Using our notation, this comultiplication is given by $(e_I)^S = (e_I)_S = e_S$. The antipode is given by $s(I)(e_I) = (-1)^{\#I} e_I$.

H2. Fix a finite set I and a decomposition (S,T) of I. If ℓ_1 and ℓ_2 are linear orders on S and T respectively, their *concatenation* $\ell_1 \cdot \ell_2$ is the unique linear order on I that restricts to ℓ_1 in S and to ℓ_2 in T, and such that $s < t$ if $s \in S$ and $t \in T$; this operation is in general not commutative. If ℓ is a linear order on I, write $\ell|_S$ for the restriction of ℓ to S, and $\bar{\ell}$ for the reverse order to ℓ. The species of linear orders L admits a bimonoid structure such that

- multiplication is given by concatenation: $\mu(S,T)(\ell_1, \ell_2) = \ell_1 \cdot \ell_2$,

- comultiplication is given by restriction: $\Delta(S,T)(\ell) = \ell|_S \otimes \ell|_T$.

In particular, this endows L with a cosymmetric bicomodule structure over $\mathbf{e}$. The map $L \longrightarrow \mathbf{e}$ that sends a linear order on a finite set I to $*_I = \{I\}$ is a map of bimonoids. The antipode is given, up to sign, by taking the reverse of a linear order: $s(I)(\ell) = (-1)^{\#I}\bar{\ell}$.

H3. If (S,T) is a decomposition of a finite set I, and $F = (F_1,\dots,F_s)$ and $G = (G_1,\dots,G_t)$ are compositions of S and of T, respectively, the *concatenation* $F\cdot G$ is the composition $(F_1,\dots,F_s,G_1,\dots,G_t)$ of I. If $F = (F_1,\dots,F_t)$ is a composition of I, the restriction of F to S is the composition $F|_S$ of S obtained from the decomposition $(F_1\cap S,\dots,F_t\cap S)$ of S by deleting empty blocks, which usually has shorter length than that of F. Finally, the reverse of a composition F is the composition $\bar{F}$ whose blocks are listed in the reverse order of those in F. The species Σ of compositions has a bimonoid structure such that

- multiplication is given by concatenation: $\mu(S,T)(F,G) = F\cdot G$,
- comultiplication is given by restriction: $\Delta(S,T)(F) = F|_S \otimes F|_T$.

This is cocommutative but not commutative. The morphism $L \longrightarrow \Sigma$ that sends a linear order $i_1\cdots i_t$ on a set I to the composition $(\{i_1\},\dots,\{i_t\})$ is a map of bimonoids. The formula for the antipode is not as immediate as the previous ones. For details, see [**AM2013**, §11].

H4. If (S,T) is a decomposition of a finite set I, and X and Y are partitions of S and T, respectively, the union $X\cup Y$ is a partition of I. If X is a partition of I, then $X|_S = \{x\cap S : x\in X\}\smallsetminus\{\varnothing\}$ is a partition of S, which we call the restriction of X to S. The species Π of partitions admits a bimonoid structure such that

- multiplication is given by the union of partitions: $\mu(S,T)(X,Y) = X\cup Y$,
- comultiplication is given by restriction: $\Delta(S,T)(X) = X|_S \otimes Y|_T$.

This is both commutative and cocommutative. The map $\Sigma \longrightarrow \Pi$ that sends a decomposition F of a set I to the partition X of I consisting of the blocks of F is a bimonoid map. The morphism $\mathbf{e} \longrightarrow \Pi$ that sends $*_I = \{I\}$ to the partition of I into singletons is also a map of bimonoids. See [**AM2013**, Theorem 33] for a formula for the antipode of Π.

H5. If p is a poset with underlying set I, and (S,T) is a decomposition of I, we say *S is a lower set of T with respect to p* and write $S \prec_p T$ if no element of T is less than an element of S for the order p, and we write p_S for $p\cap(S\times S)$, the *restriction of p to S*. The linearized species Pos of posets admits a bimonoid structure so that

- multiplication is given by the disjoint union of posets: if p^1 and p^2 are posets with underlying sets S and T, respectively, $\mu(S,T)(p^1,p^2) = p^1 \sqcup p^2$,
- comultiplication is obtained by lower sets and by restriction: for p a poset defined on $S \sqcup T$, we set $\Delta(S,T)(p) = p_S \otimes p_T$ if $S \prec_p T$, and set $\Delta(S,T)(p) = 0$ if not.

The verification that these are compatible operations making Pos into a bimonoid is straightforward. Remark this is an example of a non-commutative comultiplication, as opposed to the previous examples we gave.

It is worthwhile to remark that one can define another multiplication on this species: if p^1 and p^2 are posets on disjoint sets S and T, respectively, let $p^1 * p^2$ be the usual join of posets. This is associative and has unit the empty poset, and the inclusion of linear orders into posets $L \longrightarrow P$ is a morphisms of monoids if L is given the concatenation product. We also remark that the maps described above fit into a commutative diagram of Hopf monoids as illustrated in the figure

$$\begin{tikzcd} & \Sigma \arrow[dr] & \\ L \arrow[ur] \arrow[dr] & & \Pi \\ & \mathbf{e} \arrow[ur] & \end{tikzcd}$$

and we will analyse the resulting maps in cohomology in Chapter III. For more examples of Hopf monoids in species, and their relation to classical combinatorial results, we refer the reader to [**AM2010**, Chapter 13].

6. Monoidal structure on the category of bicomodules

Fix a comonoid c in an abelian monoidal category C, and two c-bicomodules x, x' with corresponding structure maps (λ, ρ) and (λ', ρ'). Further, assume that $c \otimes ?$ and $? \otimes c$ are exact. We define the *cotensor product* of x and x', and write $x \square x'$, for the kernel of the map $1 \otimes \lambda' - \rho \otimes 1$, as illustrated in the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & x \square x' \arrow[r] & x \otimes x \arrow[r, shift left, "1\otimes\lambda'"] \arrow[r, shift right, "\rho\otimes 1"'] & x \otimes c \otimes x' \end{tikzcd}$$

It is not immediate that $x \square x'$ is a again a c-bicomodule, and this is in fact not always true. In our case, however, the assumption that $c\otimes ?$ and $? \otimes c$ are exact renders the diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & x \,\square\, x' \arrow[r] \arrow[d, dashed] & x \otimes x' \arrow[r] \arrow[d, "\lambda\otimes 1\circ 1\otimes\rho"] & x \otimes c \otimes x' \arrow[d, "\lambda\otimes 1\circ 1\otimes\rho"] \\
0 \arrow[r] & c \otimes (x \,\square\, x') \otimes c \arrow[r] & c \otimes x \otimes x' \otimes c \arrow[r] & c \otimes x \otimes c \otimes x' \otimes c
\end{tikzcd}$$

that induces the desired map, and compatibility of the corresponding diagrams is inherited from those for $x \otimes x'$. The dotted arrow is natural, so this induces a bifunctor $?\square? : {}^c\mathrm{mod}^c \longrightarrow {}^c\mathrm{mod}^c$. It is easy to check that $x \,\square\, c = x \otimes e \simeq x$ and that $c \square x = e \otimes x \simeq x$, and that $\square$ is associative via the original associator of C. Thus we have endowed the category of c-bicomodules with a monoidal structure $(\square, c, a, r, l)$.

7. The simplicial category

The *augmented simplicial category* Δ' has as objects the finite sets $[n] = \{0, \ldots, n-1\}$ for $n \in \mathbb{N}_0$, so that in particular $[0] = \varnothing$, and arrows the monotone non-decreasing functions. The *sum* of $[n]$ and $[m]$ is $[n+m]$ and we denote it by $[n]+[m]$, and for any pair of arrows $f : [n] \longrightarrow [n]'$ and $g : [m] \longrightarrow [m]'$ we define $f+g$ by *juxtaposition*: $f+g$ takes the value $f(i)$ for $i \in \{1, \ldots, n\}$ and the value $n' + g(i-n)$ for $i \in \{n+1, \ldots, m+n\}$. The empty set $[0]$ is initial, the set $[1]$ is terminal, and addition $+ : \Delta' \times \Delta' \longrightarrow \Delta'$ makes $(\Delta', +, [0])$ into a strict monoidal category. The only arrow $\mu : [1]+[1] \to [1]$ and the only arrow $\eta : [0] \to [1]$ make $[1]$ into a monoid. The *simplicial category* Δ is the full subcategory of Δ' whose objects are the nonempty objects of Δ'. It is convenient to use the notation $[\![n]\!]$ for the set $[n+1]$.

We single out an important collection of arrows in Δ: for each $i \in \{0, \cdots, n\}$, we let $\partial_i : [\![n-1]\!] \longrightarrow [\![n]\!]$ be the unique injective monotone function whose image misses i, and dually for each $i \in \{0, \ldots, n\}$ we let $\sigma_i : [\![n+1]\!] \longrightarrow [\![n]\!]$ be the unique surjective monotone function which sends i and $i+1$ to i. We can write these arrows using the addition of Δ' and the monoid $[1]$:

$$\begin{aligned}
\partial_i &= 1^i + \eta + 1^{n+1-i} : [\![n-1]\!] \longrightarrow [\![n]\!] \\
\sigma_i &= 1^i + \mu + 1^{n+1-i} : [\![n+1]\!] \longrightarrow [\![n]\!]
\end{aligned}$$

One can visualize this category by assigning to $[\![n]\!]$ the geometrical n-simplex obtained from the convex hull of $n+1$ affinely independent points in space, and to the face and degeneracy maps the usual inclusions opposite to vertex i and the collapsing of the edge joining i and $i+1$ on such simplices. In fact, this construction gives a functor $\Delta' \to \mathsf{Top}$, and exhibits Δ' as a subcategory of Top.

DEFINITION 7.1. A *simplicial object* on a category C is a functor $A : \Delta^{\mathrm{op}} \to \mathsf{C}$; dually, a *cosimplicial object* on C is a functor $C : \Delta \to \mathsf{C}$. An *augmented simplicial object* on a category C is a functor $A : \Delta'^{\mathrm{op}} \to \mathsf{C}$, an *augmented cosimplicial object* on a category C is a functor $C : \Delta' \to \mathsf{C}$. This defines respective categories $\mathsf{Simp}(\mathsf{C})$, $\mathsf{Simp}'(\mathsf{C})$, $\mathsf{Cosimp}(\mathsf{C})$, $\mathsf{Cosimp}'(\mathsf{C})$.

The *semisimplicial category* Δ_s is the subcategory of Δ with the same objects but only the injective functions as morphisms. One can define *semisimplicial objects* and *semicosimplicial objects*, and these suffice for many homological purposes. Degeneracies give a richer homological and combinatorial structure, and will allow us to prove certain complexes are exact in particular situations. Remark that semisimplicial (but not simplicial) objects arise naturally in the cohomology of algebras which are not necessarily unital.

The following pair of lemmas gives a concrete description of arrows in the simplicial category.

LEMMA 7.2. [**Wei1994**, Exercise 8.1.1, Lemma 8.1.2] *The category Δ' is generated by the face and degeneracy maps subject to the following* cosimplicial relations*:*

$$\begin{aligned} \partial_j \partial_i &= \partial_i \partial_{j-1}, \quad i < j; \\ \sigma_j \sigma_i &= \sigma_i \sigma_{j+1}, \quad i \leqslant j; \end{aligned}$$

$$\sigma_j \partial_i = \begin{cases} \partial_i \sigma_{j-1}, & \text{if } i < j; \\ \mathrm{id}, & \text{if } i = j \text{ or } i = j+1 \\ \partial_{i-1} \sigma_j, & \text{if } i > j+1. \end{cases}$$

Moreover, every arrow $\alpha : [\![n]\!] \longrightarrow [\![m]\!]$ in Δ admits a unique epi-monic factorization $\alpha = \eta\varepsilon$ where the monic ε is uniquely a composition of face maps $\partial_{i_1} \cdots \partial_{i_s}$ with $0 \leqslant i_s \leqslant \cdots \leqslant i_1 \leqslant m$ and the epi η is uniquely a composition of degeneracy maps $\sigma_{j_1} \cdots \sigma_{j_t}$ with $0 \leqslant j_1 \leqslant \cdots \leqslant j_t \leqslant n$.

The relations obtained by composing the arrows in the reverse order give the so called *cosimplicial relations*. The description of Δ' given in these lemmas translates to the following concrete description of cosimplicial objects:

PROPOSITION 7.3. [**Wei1994**, Corollary 8.1.4] *To give a cosimplicial object X in a category C it is necessary and sufficent to give a sequence of objects $X^0, X^1, X^2, \ldots$ in C,* *together with codegeneracy and coface arrows* $d^i : X^{n-1} \longrightarrow X^n$, $s^j : X^{n+1} \longrightarrow X^n$ *that satisfy the cosimplicial relations.*

There is a dual description of simplicial objects. In light of the previous proposition, we will denote a simplicial or cosimplicial object by (X,d,s). Accordingly, a morphism $f : X \longrightarrow Y$ of simplicial objects is specified by giving arrows $f_n : X_n \longrightarrow Y_n$ in C that commute with the face and degeneracy maps. A consequence of this last proposition is that to specify an augmencted simplicial object amounts to giving a simplicial object X, an object $X_{-1} \in \mathsf{C}$ and an arrow $\varepsilon : X_0 \to X_{-1}$ such that $\varepsilon\partial_0 = \varepsilon\partial_1$. Analogous remarks hold for semisimplicial, semicosimplicial and cosimplicial objects.

DEFINITION 7.4. Let (X,d) and (Y,δ) be semisimplicial objects over a category, and let $f,g : X \longrightarrow Y$ be simplicial morphisms. A *semisimplicial homotopy* h between f and g, denoted by $h : f \simeq g$, is a sequence of arrows $h_i : X_n \longrightarrow Y_{n+1}$, one for each $0 \leqslant i \leqslant n$ and each $n \geqslant 0$, such that $\delta_0 h_0 = f, \delta_{n+1} h_n = g$ and

$$\delta_i h_j = \begin{cases} h_{j-1} d_i & i < j, \\ \delta_i h_{i-1} & i = j, 0 < i \leqslant n, \\ h_j d_{i-1} & i > j+1. \end{cases}$$

If there is a semisimplicial homotopy $h : f \simeq g$ we say that f and g are *semisimplicially homotopic.* One can define *simplicial homotopies* by extending the above list of relations to include relations involving degeneracies.

Fix an abelian category C, and write $\mathsf{Ch}_+(\mathsf{C})$ for the category of bounded below chain complexes over C. There is a functor $\mathsf{Semisimp}(\mathsf{C}) \longrightarrow \mathsf{Ch}_+(\mathsf{C})$ obtained as follows. If (X,∂) is a semisimplicial object over C, the *associated chain complex* (CX,d) of X has components those of X and differential $d = \sum_{i=0}^n (-1)^i \partial_i : X_n \longrightarrow X_{n-1}$. The composition $d \circ d$ is zero: one can split the double sum on indices i and j at those for which $i < j$ and at those for which $i \geqslant j$, and the terms then cancel out in pairs $(-1)^{i+j}\partial_i\partial_j$ and $(-1)^{i+j-1}\partial_{j-1}\partial_i$ by virtue of the simplicial identities. It follows that (CX,d) is indeed a complex. If $f : X \longrightarrow X'$ is a morphism of semisimplicial objects there is induced a morphism of complexes $Cf : CX \longrightarrow CX'$ that coincides with f componentwise. Moreover, homotopies are sent to homotopies:

LEMMA 7.5. *If two semisimplical maps are semisimplicially homotopic, the induced maps on the associated chain complexes are chain homotopic.*

PROOF. Let $f, g : X \longrightarrow Y$ be simplicial maps with $h : f \simeq g$ a simplical homotopy. Define arrows $H_n : X_n \longrightarrow Y_{n+1}$ by $H = \sum_{i=0}^{n} (-1)^i h_i$. When computing $\delta H + Hd$ the terms $\delta_i h_i$ and $\delta_i h_{i-1}$ cancel, the terms $\delta_i h_j$ and $h_{j-1}\delta_i$ cancel when $i < j$, the terms $\delta_i h_j$ and $h_j d_{i-1}$ cancel when $i > j+1$ and all that is left is $\delta_0 h_0 - \delta_{n+1} h_n = f - g$. ◀

If X is a simplicial object over C, the *homology of* X is the graded object $H(X) = H(CX, d)$, and we say that X is *acyclic* if $H(X) = 0$.

For a cosimplicial object (X, ∂, σ) in an abelian category, we define DX^n to be the subobject $\sum \sigma^j (X^{n+1})$. This defines a subcomplex DX, the *subcomplex of degenerate simplices* in X; to see this, note that when computing $d\sigma^j$ in terms of face maps, the only terms that do not involve degeneracies cancel at $i = j, j+1$. Along with DX, we can consider the complex (NX, δ) with $NX^n = \bigcap_{i=0}^{n-1} \ker(\partial^i : X^n \longrightarrow X^{n+1})$ and differential $\delta^n = (-1)^n \partial^n$, which we call the *Moore* or *normalized* subcomplex of CX. The following lemma will simplify many calculations in what follows. We write $\bar{C}X$ for the quotient CX/DX.

LEMMA 7.6. [**Wei1994**, Lemma 8.3.7, Theorem 8.3.8] *For any cosimplicial object X over an abelian category, DX is acyclic and $NX \oplus DX = CX$, so that the inclusion $NX \hookrightarrow CX$ and the projection $CX \longrightarrow \bar{C}X$ are quasi-isomorphisms.*

8. Cohomology in abelian monoidal categories

We now give a general construction that takes a comonoid (c, Δ, ε) on an abelian monoidal category C and defines a cohomological functor on the category ${}^c\mathsf{mod}^c$ of c-bicomodules. To do this, we need the following observation, which dualizes the usual property of free modules: there is a functor $\mathsf{C} \to {}^c\mathsf{mod}^c$ that sends an object y in C to $c \otimes y \otimes c$ with the natural bicomodule structure defined by Δ, and acts in the obvious way on arrows. The bicomodules in the image of this functor are "free"; this is stated more precisely in the following lemma.

LEMMA 8.1. *Let x be a c-bicomodule and let $y \in \mathsf{C}$. There is a natural isomorphism*

$$\eta : \hom_{{}^c\mathsf{mod}^c}(x, c \otimes y \otimes c) \longrightarrow \hom_{\mathsf{C}}(x, y)$$

such that if $\alpha : x \longrightarrow c \otimes y \otimes c$ is a morphism of c-bicomodules and $\beta : x \longrightarrow y$ is a morphism in C, the following diagrams commute:

$$\begin{tikzcd} y & 1\otimes y \arrow[l, "l"'] & 1\otimes y\otimes 1 \arrow[l, "1\otimes r"'] \\ x \arrow[u, "\eta(\alpha)"] \arrow[rr, "\alpha"] & & c\otimes y\otimes c \arrow[u, "\varepsilon\otimes 1\otimes\varepsilon"'] \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, "\lambda"] \arrow[d, "\eta^{-1}(\beta)"] & c\otimes x \arrow[d, "1\otimes\rho"] \\ c\otimes y\otimes c & c\otimes x\otimes c. \arrow[l, "1\otimes\beta\otimes 1"] \end{tikzcd}$$

PROOF. Suppose that $\beta : x \longrightarrow y$ is a morphism in C. The arrow $\eta^{-1}(\beta)$ uniquely defined by the commutativity of the rightmost diagram is a morphism of bicomodules: this follows directly from the fact that λ and ρ are compatible coactions. Indeed, this compatibility means that —writing χ for the composition $1\otimes\rho\circ\lambda : x \longrightarrow c\otimes x\otimes c$— we have the equality $\Delta\otimes 1\otimes\Delta\circ\chi = 1\otimes\chi\otimes 1\circ\chi$, and then

$$\begin{aligned} \Delta\otimes 1\otimes\Delta\circ 1\otimes\beta\otimes 1\circ\chi &= 1\otimes\beta\otimes 1\circ\Delta\otimes 1\otimes\Delta\circ\chi \\ &= 1\otimes\beta\otimes 1\circ 1\otimes\chi\otimes 1\circ\chi, \end{aligned}$$

as desired. On the other hand, if $\alpha : x \longrightarrow c\otimes y\otimes c$ is a morphism of c-bicomodules, then $\eta(\alpha) : x \longrightarrow y$ is *a fortiori* a morphism in C. That η and η^{-1} are inverse bijections follows from the equality

$$\varepsilon\otimes 1\otimes\varepsilon\circ\chi = l^{-1}\otimes 1\otimes r^{-1},$$

and this is what the lemma claims. ◀

Let (c, Δ, ε) be a comonoid in C, fixed throughout what remains of this section, and let us construct a cosimplicial c-bicomodule $\Omega(c) : \Delta' \longrightarrow {}^c\mathsf{mod}^c$, which we refer to the *cobar construction on c*. For each nonnegative integer n, we put $\Omega(c)^n = c^{\otimes(n+2)}$, and define coface and codegeneracy maps

$$\begin{aligned} \partial^i &= 1^{\otimes i}\otimes\Delta\otimes 1^{\otimes(n+1-i)} : \Omega(c)^n \longrightarrow \Omega(c)^{n+1} \quad \text{for } 0\leqslant i\leqslant n, \\ \sigma^j &= 1^{\otimes j+1}\otimes\varepsilon\otimes 1^{\otimes(n-j+1)} : \Omega(c)^{n+1} \longrightarrow \Omega(c)^n \quad \text{for } 0\leqslant j\leqslant n. \end{aligned}$$

It is important to notice these coface and codegeneracy arrows are all morphisms of c-bicomodules —this can be seen as a special case of Lemma 8.1.

We now want to show that $\Omega(c)$ provides us with a resolution of c in the category of c-bicomodules, with augmentation given by the comultiplication map Δ. In other words, that

PROPOSITION 8.2. *The complex associated to $\Omega(c)$ has trivial homology in positive degrees, and $\Delta : c \longrightarrow c\otimes c = \Omega(c)^0$ induces an isomorphism $c \longrightarrow H^0(\Omega(c))$.*

PROOF. Let $U : {}^c\mathsf{mod}^c \longrightarrow \mathsf{C}$ be the forgetful functor, and write $\Omega(c)' = U\circ\Omega(c)$. Since U reflects exactness, it suffices to prove the analogous claims for $\Omega(c)'$, which is a cosimplicial object in C. To this end, we consider the complex

$$0 \longrightarrow c \xrightarrow{\Delta} \Omega(c)'^0 \xrightarrow{d^0} \Omega(c)'^1 \xrightarrow{d^1} \cdots$$

which has c in degree -1, and show that it is exact. In fact, it is contractible: there is a contracting homotopy H with components

$$H^n = 1^{\otimes n+2}\otimes\varepsilon : \Omega(c)'^{n+1} \longrightarrow \Omega(c)'^n$$

for $n \geqslant 1$ and $H^0 = 1\otimes\varepsilon : \Omega(c)'^0 \longrightarrow c$. Indeed, a direct computation shows that if ∂^i is the ith coface component of d, then $H^n\partial^i = \partial^{i-1}H^{n-1}$ if $0 \leqslant i < n$ and $H^n\partial^n = \mathrm{id}$, so that $Hd + dH = \mathrm{id}$. ◀

If x is a c-bicomodule we obtain, by applying the functor $\hom(x, ?)$, omitting the subscript ${}^c\mathsf{mod}^c$ to lighten up the notation, a cosimplicial k-module $\hom(x,\Omega(c))$, with associated cochain complex

$$0 \longrightarrow \hom(x, c^{\otimes 2}) \xrightarrow{\delta^0} \hom(x, c^{\otimes 3}) \xrightarrow{\delta^1} \hom(x, c^{\otimes 4}) \xrightarrow{\delta^2} \cdots$$

and differential acting by precomposition. The *cohomology $H^*(x, c)$ of c with values in x* is the cohomology of this complex.

Using the identifications provided in Lemma 8.1 we can give an alternative description of this cochain complex, which has the advantage of involving hom-spaces in the underlying category C instead of those in ${}^c\mathsf{mod}^c$.

THEOREM II.8.3. *Let x be a c-bicomodule with left and right coactions λ and ρ. There is a cosimplicial k-module $C^*(x, c)$ with*

C1. *components $C^n(x,c) = \hom_{\mathsf{C}}(x, c^{\otimes n})$ for each $n \geqslant 0$,*

C2. *coface maps* $\partial^i : C^n(x,c) \longrightarrow C^{n+1}(x,c)$*, for* $i \in \{0,\cdots,n+1\}$*, such that*

$$\partial^i(f) = \begin{cases} (1\otimes f)\circ\lambda & \text{if } i=0,\\ (1^{\otimes i-1}\otimes\Delta\otimes 1^{\otimes n-i})\circ f & \text{if } 0<i<n+1,\\ (f\otimes 1)\circ\rho & \text{if } i=n+1 \end{cases}$$

for each morphism $f : x \longrightarrow c^{\otimes n}$ *in* C*, and*

C3. *codegeneracy maps* $\sigma^j : C^{n+1}(x,c) \to C^n(x,c)$*, for each* $j \in \{0,\ldots,n+1\}$*, given by* $\sigma^j(f) = (1^{\otimes j}\otimes\varepsilon\otimes 1^{\otimes n-j})\circ f$ *for each morphism* $f : x \longrightarrow c^{\otimes n+1}$ *in* C*,*

and a functorial isomorphism of cosimplicial k*-modules* $\Psi : \hom(x,\Omega(c)) \longrightarrow C^*(x,c)$.

As a consequence of this, the complex associated to $C^*(x,c)$, namely

$$0 \longrightarrow \hom_{\mathsf{C}}(x,1) \xrightarrow{d^0} \hom_{\mathsf{C}}(x,c) \xrightarrow{d^1} \hom_{\mathsf{C}}(x,c^{\otimes 2}) \xrightarrow{d^2} \cdots$$

computes the cohomology of c with values in x.

PROOF. For each $n \geqslant 0$, let $\Psi^n : \hom(x,\Omega(c)^n) \longrightarrow C^n(x,c)$ be given by $\Psi^n(f) = \eta_{x,c^n}(f)$, where we identify $c^{\otimes(n+2)} = c\otimes c^{\otimes n}\otimes c$. That this is an isomorphism follows from Lemma 8.1, and we now check that the following diagrams are commutative

$$\begin{tikzcd} \hom(x,\Omega(c)^n) \arrow[r, "\partial^i"] \arrow[d, "\Psi^n"] & \hom(x,\Omega(c)^{n+1}) \arrow[d, "\Psi^{n+1}"] \\ C^n(x,c) \arrow[r, "\partial^i"] & C^{n+1}(x,c) \end{tikzcd}$$

$$\begin{tikzcd} \hom(x,\Omega(c)^n) \arrow[d, "\Psi^n"] & \hom(x,\Omega(c)^{n+1}) \arrow[l, "\sigma^j"'] \arrow[d, "\Psi^{n+1}"] \\ C^n(x,c) & C^{n+1}(x,c) \arrow[l, "\sigma^j"'] \end{tikzcd}$$

for each $i \in \{0,\ldots,n\}$ and each $j \in \{0,\ldots,n+1\}$. Indeed, for $f : x \longrightarrow c^{\otimes n+2}$, we have, omitting the unitors,

$$\begin{aligned} \Psi^{n+1}\partial^i(f) &= \varepsilon\otimes 1\otimes\varepsilon\circ 1^{\otimes i}\otimes\Delta\otimes 1^{\otimes(n-i+1)}\circ f \\ &= 1^{\otimes i-1}\otimes\Delta\otimes 1^{\otimes(n-i)}\circ\varepsilon\otimes 1\otimes\varepsilon\circ f \\ &= \partial^i\Psi^n(f), \end{aligned}$$

if $0 < i < n$. To see the diagram commutes for for $i = 0$, we will instead compute $\partial^0(\Psi^n)^{-1}(g)$ for $g : x \longrightarrow c^{\otimes n}$ a morphism in C, and check this equals $(\Psi^{n+1})^{-1}\partial^0(g)$:

$$
\begin{aligned}
\partial^0(\Psi^n)^{-1}(g) &= \Delta \otimes 1^{n+1} \circ 1 \otimes g \otimes 1 \circ \lambda \otimes 1 \circ \rho \\
&= 1 \otimes g \otimes 1 \circ \Delta \otimes 1 \otimes 1 \circ \lambda \otimes 1 \circ \rho \\
&= 1 \otimes g \otimes 1 \circ (1 \otimes \lambda \circ \lambda) \otimes 1 \circ \rho \\
&= 1 \otimes \circ (g \otimes 1 \circ \lambda) \otimes 1 \circ \lambda \otimes 1 \circ \rho \\
&= (\Psi^{n+1})^{-1}\partial^0(g).
\end{aligned}
$$

The case $i = n$ is dual to this. We omit the verification that the diagrams involving codegeneracies commute, which are equally straightforward. It follows the collection $\Psi = (\Psi^n)$ is, in fact, an isomorphism of cosimplicial k-modules, as claimed. ◀

CHAPTER III

The cohomology of combinatorial species

1. Definitions and first examples

Let $\mathbf{c}$ be a comonoid in Sp_k and $\mathbf{x}$ a $\mathbf{c}$-bicomodule. The *Cartier cohomology of* $\mathbf{c}$ *with values in* $\mathbf{x}$ is the cohomology of the cosimplicial k-module $C^*(\mathbf{x},\mathbf{c})$ constructed in Theorem II.8.3, and we denote it by $H^*(\mathbf{x},\mathbf{c})$. As usual, we will write $HH^*(\mathbf{c})$ for $H^*(\mathbf{c},\mathbf{c})$. In the following we will mainly consider the case in which $\mathbf{c}$ is the exponential species, but will make it clear when a certain result can be extended to other comonoids. Usually, it will be necessary that $\mathbf{c}$ is linearized and with a linearized bimonoid structure, and we will usually require that $\mathbf{c}$ be connected. Because of the plethora of relevant examples of such bimonoids found in [**AM2010**] and other articles by the same authors, such as [**AM2013**], there is no harm in restricting ourselves to such species.

Fix an $\mathbf{e}$-bicomodule $\mathbf{x}$. The complex $C^*(\mathbf{x},\mathbf{e})$, which we will denote more simply by $C^*(\mathbf{x})$, has in degree q the collection of morphisms of species $\alpha : \mathbf{x} \longrightarrow \mathbf{e}^{\otimes q}$. Such a morphism is determined by a collection of k-linear maps $\alpha(I) : \mathbf{x}(I) \longrightarrow \mathbf{e}^{\otimes q}(I)$, one for each finite set I, which is equivariant, in the sense that for each bijection $\sigma : I \longrightarrow J$ between finite sets, and every $z \in \mathbf{x}(I)$, the equality $\sigma(\alpha(I)(z)) = \alpha(J)(\sigma z)$ holds.

Now observe that $\mathbf{e}^{\otimes q}(I)$ is a free k-module with basis the tensors of the form $F_1 \otimes \cdots \otimes F_q$ with $(F_1,\ldots,F_q)$ a decomposition of I; for simplicity, we use the latter notation for such basis elements. In terms of this basis, we can write

$$\alpha(I)(z) = \sum_{F \vdash_q I} \alpha(F)(z)\, F$$

where $\alpha(F)(z) \in k$.

As described in Chapter II, Section 4, the cochain α is completely determined by an equivariant collection of functionals $\alpha(F) : \mathbf{x}(I) \longrightarrow k$, the components of α, one for each decomposition F of a finite set I. The equivariance condition is now that, for a bijection $\sigma : I \longrightarrow J$, and $(F_1,\ldots,F_q)$ a decomposition of I, we have

$$\alpha(F_1,\ldots,F_q)(z) = \alpha(\sigma(F_1),\ldots,\sigma(F_q))(\sigma z)$$

for each $z \in \mathbf{x}(I)$. Recall that when writting $\alpha(F)(z)$ we omit I, recalling that it is always the case $I = \cup F$.

Now fix a q-cochain $\alpha : \mathbf{x} \longrightarrow \mathbf{e}^{\otimes q}$ in $C^*(\mathbf{x})$. By the remarks in the last paragraph, to determine the $(q+1)$-cochain $d\alpha : \mathbf{x} \longrightarrow \mathbf{e}^{\otimes(q+1)}$ it is enough to determine its components. If we pick a decomposition $F = (F_0, \ldots, F_q)$ of a set I, then the component of the ith coface $\partial^i \alpha$ at F is given, for $z \in X(I)$, by

$$(\partial^i \alpha)(F_0, \ldots, F_q)(z) = \begin{cases} \alpha(F_1, \ldots, F_q)(z /\!\!/ F_0^c) & \text{if } i = 0, \\ \alpha(F_0, \ldots, F_i \cup F_{i+1}, \ldots, F_{q+1})(z) & \text{if } 0 < i < q+1, \\ \alpha(F_0, \ldots, F_{q-1})(z \setminus\!\!\setminus F_q^c) & \text{if } i = q+1. \end{cases} \tag{2}$$

Indeed, let us follow the construction carried out in Theorem II.8.3, and compute each coface map explicitly. If $z \in X(I)$, to compute $\partial^0 \alpha(z)$, we must coact on z to the left and evaluate the result at α, that is

$$(1 \otimes \alpha \circ \lambda)(I)(z) = \sum_{(S,T) \vdash I} *_S \otimes \alpha(T)(z /\!\!/ T),$$

and the coefficient at a decomposition $F = (F_0, \ldots, F_q)$ is $\alpha(F_1, \ldots, F_q)(z /\!\!/ F_0^c)$. The same argument gives the last coface map. Now consider $0 < i < q+1$, so that we must take $z \in \mathbf{x}(I)$, apply α, and then comultiply the result at coordinate i. Concretely, write

$$\alpha(I)(z) = \sum_{F \vdash_q I} \alpha(I)(F)(z) F$$

and pick a decomposition $F' = (F_0, \ldots, F_q)$ into $q+1$ blocks of I. There exists then a unique $F \vdash_q I$ such that $1^{i-1} \otimes \Delta \otimes 1^{q-i}(F) = F'$, to wit, $F = (F_0, \ldots, F_i \cup F_{i+1}, \ldots, F_q)$, and in this way we obtain the formulas of Equation (2).

The codegeneracy maps are easier to describe: they are obtained by inserting an empty block into a decomposition. Concretely, for each $j \in \{0, \ldots, q+1\}$,

$$(\sigma^j \alpha)(F_1, \ldots, F_q)(z) = \alpha(F_1, \ldots, F_j, \varnothing, F_{j+1}, \ldots, F_q).$$

As a consequence of this, a cochain $\alpha : \mathbf{x} \longrightarrow \mathbf{e}^{\otimes q}$ in $C^*(\mathbf{x})$ is in the normalized subcomplex $\overline{C}^*(\mathbf{x})$ if its components are such that $\alpha(F)(z) = 0 \in k$ whenever F contains an empty block. Alternatively, we can construct a comonoid $\bar{\mathbf{e}}$ with $\bar{\mathbf{e}}(\varnothing) = 0$ and

$\bar{\mathbf{e}}(I) = \mathbf{e}(I)$ whenever I is nonempty, and describe the normalized complex $\overline{C}^*(\mathbf{x})$ as the complex of maps $\mathbf{x} \longrightarrow \bar{\mathbf{e}}^{\otimes *}$ with differential induced by the alternating sum of the coface maps we just described. Remark that $\bar{\mathbf{e}}^{\otimes q}(I)$ has basis the *compositions* of I into q blocks, while $\mathbf{e}^{\otimes q}(I)$ has basis the *decompositions* of I into q blocks. In particular, $\bar{\mathbf{e}}^{\otimes q}(I) = 0$ if $q > \#I$, while $\mathbf{e}^{\otimes q}(I)$ is always nonzero. This observation will be crucial in Chapter IV.

With this at hand, let us now look at low dimensional cohomology groups, starting with the 0-cocycles.

PROPOSITION 1.1. *Let $\mathbf{x}$ be an $\mathbf{e}$-bicomodule. Then $H^0(\mathbf{x})$ is isomorphic to the kernel of the map*

$$\vartheta_0 : \hom_k(\mathbf{x}(\varnothing), k) \longrightarrow \hom_k(\mathbf{x}([1]), k)$$

such that for $f : \mathbf{x}(\varnothing) \longrightarrow k$, we have $\vartheta_0(f)(z) = f(z /\!\!/ \varnothing) - f(z \backslash\!\!\backslash \varnothing)$.

PROOF. The space $C^0(\mathbf{x})$ is that of maps $\alpha : \mathbf{x} \longrightarrow \mathbf{1}$. Since $\mathbf{1}$ is concentrated in cardinal zero, such a things amounts to a k-linear map $\alpha(\varnothing) : \mathbf{x}(\varnothing) \longrightarrow k$. Moreover, the coboundary $d\alpha$ is determined by its values at decompositions of I into one block, of which there is only one, namely (I). By definition,

$$(d\alpha)(I)(z) = \alpha(\varnothing)(z /\!\!/ \varnothing) - \alpha(\varnothing)(z \backslash\!\!\backslash \varnothing), \tag{3}$$

and, in particular, by taking I to be the set $[1]$, we see that $\vartheta(\alpha(\varnothing)) = 0$. This shows that the map $\alpha \in C^0(\mathbf{x}) \longmapsto \alpha(\varnothing) \in \hom_k(\mathbf{x}(\varnothing), k)$, which is a bijection, restricts to a map $H^0(\mathbf{x}) \longrightarrow \ker \vartheta_0$. Let us show this is bijective, and, to do this, that it is onto. Pick $f \in \ker \vartheta_0$, and let $\alpha \in C^0(\mathbf{x})$ be the unique 0-cochain such that $\alpha(\varnothing) = f$. We have to show it is a cocycle, namely, that the right hand side of Equation (3) vanishes for any set I, and every $z \in X(I)$. We do this by induction on the cardinal of I. If I is empty, there is nothing to do. If not, choose $i \in I$ and observe that

$$\begin{aligned}
\alpha(z \backslash\!\!\backslash \varnothing) &= \alpha(z \backslash\!\!\backslash I \smallsetminus i \backslash\!\!\backslash \varnothing) \\
&= \alpha(z \backslash\!\!\backslash I \smallsetminus i /\!\!/ \varnothing) && \text{by induction} \\
&= \alpha(z /\!\!/ \{i\} \backslash\!\!\backslash \varnothing) && \text{because } \lambda \text{ and } \rho \text{ are compatible} \\
&= \alpha(z /\!\!/ \{i\} /\!\!/ \varnothing) && \text{since } \vartheta_0(f) = 0 \\
&= \alpha(z /\!\!/ \varnothing).
\end{aligned}$$

This completes the proof of the proposition. ◀

A particular but useful consequence of the above is the following corollary.

COROLLARY 1.2. *If* $\mathbf{x}$ *is cosymmetric or linearized and connected,* $H^0(\mathbf{x})$ *is isomorphic to* $\hom_k(\mathbf{x}(\varnothing), k)$.

PROOF. In both cases, the map ϑ_0 is zero. ◀

We now consider 1-cochains, which have a similar description. For every finite set I and $i \in I$, denote by τ_i the unique bijection $\{i\} \longrightarrow [1]$. Note that for any $\mathbf{e}$-bicomodule $\mathbf{x}$ there is a 1-cocycle $\kappa : \mathbf{x} \longrightarrow E$ such that $\kappa(I)(z) = \#I$, which we call the *cardinality cocycle of* $\mathbf{x}$.

PROPOSITION 1.3. *Let* $\mathbf{x}$ *be an* $\mathbf{e}$*-bicomodule, and consider the map*

$$\vartheta_1 : \hom_k(\mathbf{x}([1]), k) \longrightarrow \hom_k(\mathbf{x}([2]), k)$$

such that for $f : \mathbf{x}([1]) \longrightarrow k$ *and* $z \in \mathbf{x}([1])$,

$$\vartheta_1(f)(z) = f(z /\!\!/ \{1\}) - f(\tau_2(z /\!\!/ \{2\})) - f(z \mathbin{\backslash\!\!\backslash} \{1\}) + f(\tau_2(z \mathbin{\backslash\!\!\backslash} \{2\})).$$

Then $\vartheta_1 \circ \vartheta_0 = 0$*, the* k*-linear map* $r^1 : \alpha \in C^1(\mathbf{x}) \longmapsto \alpha([1]) \in \hom_k(\mathbf{x}([1]), k)$ *restricts to an injection* $Z^1(\mathbf{x}) \longrightarrow \ker \vartheta_1$ *and* $r^1(B^1(\mathbf{x})) = \operatorname{im} \vartheta_0$*, so* r^1 *induces an injective map*

$$\overline{r}^1 : H^1(\mathbf{x}) \longrightarrow \frac{\ker \vartheta_1}{\operatorname{im} \vartheta_0}.$$

We will see that $\overline{r}^1$ is in fact an isomorphism in Chapter IV.

PROOF. That $\vartheta_1 \circ \vartheta_0 = 0$ can be seen by direct computation. If $\alpha : X \longrightarrow E$ is a cochain, the coboundary $d\alpha$ is such that for every decomposition (S, T) of a finite set I and every $z \in X(I)$,

$$d\alpha(S, T)(z) = \alpha(S)(z /\!\!/ S) - \alpha(I)(z) + \alpha(T)(z \mathbin{\backslash\!\!\backslash} T).$$

If α is a cocycle, this equation shows, first, that $\alpha(\varnothing) = 0$, and second, by an obvious induction, that α is completely determined by its values at sets of cardinal one. We deduce from this that r^1 restricts to an injection on $Z^1(\mathbf{x})$. Moreover, it has image in $\ker \vartheta_1$, since both $\alpha(\{1\})(z /\!\!/ \{1\}) + \alpha(\{2\})(z \mathbin{\backslash\!\!\backslash} \{2\})$ and $\alpha(\{2\})(z /\!\!/ \{2\}) + \alpha(\{1\})(z \mathbin{\backslash\!\!\backslash} \{1\})$

equal $\alpha([2])(z)$. Finally, it is immediate that $r^1(B^1(\mathbf{x})) = \operatorname{im}\vartheta_0$, so the last claim in the proposition follows. ◀

Remark that by Lemma 7.6 we may calculate $H^*(\mathbf{x})$ using the normalized complex of $C^*(\mathbf{x})$. The motivating reason is that decompositions of sets are replaced by *compositions*, and every finite set has *finitely many* compositions, while every finite set always has infinitely many decompositions.

2. First computations

The unit species $\mathbf{1}$ and the exponential species $\mathbf{e}$ offer two simple examples of the calculation of the cohomology of a species. Although these are quite simple, we record them in the form of propositions. We also consider low dimensional cohomology groups for the species of linear orders L, whose $\mathbf{e}$-bicomodule structure is defined Chapter II, Section 5.

We endow $\mathbf{1}$ with the symmetric bicomodule structure obtained from the unit morphism $\varepsilon : \mathbf{1} \longrightarrow E$ that is an isomorphism in cardinal 0.

PROPOSITION 2.1. *We have $H^0(\mathbf{1}) = k$ and $H^p(\mathbf{1}) = 0$ for $p > 0$.*

PROOF. Let us show that the normalized complex $\overline{C}^*(\mathbf{1})$ has $\overline{C}^p(\mathbf{1}) = 0$ if $p > 0$, and that $\overline{C}^0(\mathbf{1}) \simeq k$: this proves the proposition. A normalized q-cochain $\alpha : \mathbf{1} \longrightarrow E^{\otimes q}$ is determined uniquely by a scalar for each composition of $\varnothing$ into q blocks. If $q > 0$, there are no such compositions, and therefore $\overline{C}^p(\mathbf{1}) = 0$. On the other hand, if $q = 0$, there is a unique composition of $\varnothing$ into no blocks, and this yields a k-linear map $\alpha : k \simeq \mathbf{1}(\varnothing) \longrightarrow E(\varnothing) \simeq k$. ◀

This is a first toy example of how normalization simplifies computations significantly.

The species $\mathbf{e}$ is canonically an $\mathbf{e}$-bicomodule. Denote by $C^*(\mathbb{N}_0, k)$ the cosimplicial k-module that computes the semigroup cohomology of $\mathbb{N}$ with values in k. We refer the reader to [**CE1956**, Chapter VIII, §3] for details.

PROPOSITION 2.2. *There is an isomorphism of cosimplicial k-modules*

$$\Gamma : C^*(\mathbf{e}) \longrightarrow C^*(\mathbb{N}_0, k)$$

that assigns to a cochain $\alpha : \mathbf{e} \longrightarrow \mathbf{e}^{\otimes p}$ the cochain $\Gamma(\alpha) : \mathbb{N}_0^p \longrightarrow k$ such that

$$\Gamma(\alpha)(n_1, \ldots, n_p) = \alpha(F_1, \ldots, F_p)(*_I)$$

where I is any finite set of $\sum n_i$ elements and $(F_1,\dots,F_p)$ is a decomposition of I with $\#F_i = n_i$. As a consequence of this, there is an isomorphism $HH^*(\mathbf{e}) \longrightarrow H^*(\mathbb{N}_0,k)$, so that $HH^0(\mathbf{e})$ and $HH^1(\mathbf{e})$ are free of rank one, and $HH^p(\mathbf{e}) = 0$ if $p > 1$. Moreover, a generator of $HH^1(\mathbf{e})$ is the cardinality cocycle.

PROOF. Note first that Γ is well-defined because every cochain in $C^*(\mathbf{x})$ is equivariant. Indeed, if $(n_1,\dots,n_p) \in \mathbb{N}^p$, pick I and $F = (F_1,\dots,F_p)$ as in the statement of the proposition, and consider J and $G = (G_1,\dots,G_p)$ with the same properties. There is a unique bijection $\sigma : I \longrightarrow J$ such that $\sigma(F) = G$, and then, since σ sends $*_I$ to $*_J$, we have

$$\alpha(F_1,\dots,F_p)(*_I) = \alpha(G_1,\dots,G_p)(*_J).$$

An easy verification shows that Γ is a cosimplicial map, and that the inverse isomorphism is given by the map $\Lambda : C^*(\mathbb{N},k) \longrightarrow C^*(\mathbf{e})$ that assigns to a cochain $f : \mathbb{N}^p \longrightarrow k$ the cochain $\Lambda(f) : \mathbf{e} \longrightarrow \mathbf{e}^{\otimes p}$ such that

$$\Lambda(f)(F_1,\dots,F_p)(*_I) = f(\#F_1,\dots,\#F_p).$$

To conclude the proof of the proposition, we need only recall from [**CE1956**, Chapter X, §5] that $H^*(\mathbb{N},k)$ is an exterior algebra $\Lambda(x)$ with $|x| = 1$, and that, because $\mathbf{e}$ is cocommutative, the differential $d^0 : C^0(\mathbf{e}) \longrightarrow C^1(\mathbf{e})$ is zero, so the cardinality cocycle is not a coboundary. ◀

We have defined a cosymmetric $\mathbf{e}$-bicomodule structure on the species L of linear orders, and we can easily compute its zeroth and first cohomology groups.

PROPOSITION 2.3. *Both $H^0(L)$ and $H^1(L)$ are free k-modules of rank one, and the cardinality cocycle generates $H^1(L)$.*

PROOF. Since L is connected, we know that $d^0 = 0$ and that $H^0(L)$ equals the free k-module $\hom_k(L(\varnothing),k) \simeq k$. Consider now a 1-cochain $\alpha : L \longrightarrow E$. For each finite set I of n elements, the scalars $\alpha(I)(\ell)$ for $\ell \in L(I)$ are completely determined by the scalar $\alpha([n])(\mathrm{id})$. Indeed, for any choice of linear order ℓ in I there is a unique bijection $\sigma : I \longrightarrow [n]$ such that $\sigma(\ell) = \mathrm{id}$, and then $\alpha(I)(\ell) = \alpha([n])(\mathrm{id})$ by equivariance of α. The cochain α is thus uniquely determined by the sequence of scalars $(\alpha([n])(\mathrm{id}) : n \geqslant 0)$ which we view as a map $a : \mathbb{N}_0 \longrightarrow k$. Moreover, the cocycle condition for α translates to the fact that a is additive, and when it is satisfied, a is completely determined by

$a(1)$. We conclude that $H^1(L) = Z^1(L)$ is isomorphic to k, and because there are no 1-coboundaries, we see that the cardinality cocycle generates $H^1(L)$. ◀

It is worthwhile to note that we just verified, in this special case, that the map $\overline{r}^1$ of Proposition 1.3 is an isomorphism. Let us record there is a 2-cochain $L \longrightarrow \mathbf{e}^{\otimes 2}$, which we call the *Schubert cocycle* and denote by sch, which is as follows. For a linear order ℓ on a finite set I and a decomposition (S,T) of I, we set

$$\operatorname{sch}(S,T)(\ell) = \#\{(i,j) \in S \times T : i > j \text{ according to } \ell\}.$$

This cochain is defined in [**AM2010**] and it is easily verified it is, as its name indicates it, a cocycle.

3. The cup product

A *graded k-algebra*[1] is a k-algebra A endowed with a direct sum decomposition

$$A = A^0 \oplus A^1 \oplus A^2 \oplus \cdots$$

into submodules A^0, A^1, A^2, … such that for any choice of natural numbers p and q, we have $A^p A^q \subseteq A^{p+q}$, where $A^p A^q$ denotes the submodule generated by the pointwise products of elements of A^p and A^q. Every graded k-algebra is thus, in particular, a k-module, A^0 is a subring of A, and A is an A^0-module. We call A^p the homogeneous component of degree p, say a non-zero element $a \in A^p$ is homogeneous of degree p, and in that case write $|a|$ for its degree. It is clear from the definitions that if A is unital then necessarily $1 \in A^0$, in which case A^0 is unital. A k-linear map between graded vector spaces $f : A \longrightarrow B$ is *graded of degree r* if for every $p \in \mathbb{N}_0$, we have $f(A^p) \subseteq B^{p+r}$. If A is a graded k-algebra and a and b are homogeneous elements of A, their *commutator* is the element $[a,b] = ab - (-1)^{|a||b|}ba$, and this definition extends linearly to every pair of elements of A. A graded k-algebra is *commutative* if the commutator of every pair of elements vanishes. For this, it is necessary and sufficient that $ab = (-1)^{|a||b|}ba$ for every pair of homogeneous elements a and b of A.

A *differential graded algebra* is a pair (A,d) where A is a graded k-algebra and d is a k-linear endomorphism of A of degree 1 such that $d \circ d = 0$, satisfying the following

[1] We should be speaking of $\mathbb{N}_0$-graded k-algebras, but this distinction will not be necessary.

Leibniz rule: for any pair of homogeneous elements $a, b \in A$ we have

$$d(ab) = (da)b + (-1)^{|a|} a\,db.$$

Such an endomorphism $d : A \longrightarrow A$ is called a *graded derivation* of A. From the Leibniz rule it follows that $\ker d$ is a graded subalgebra of A and that $\operatorname{im} d$ is an homogeneous ideal of $\ker d$, so that we may form the graded k-algebra

$$H(A,d) = \frac{\ker d}{\operatorname{im} d}$$

which we call the *cohomology algebra of* (A,d). The homogeneous elements of $\ker d$ are called *cocycles*, and a cocycle of degree p is called a p*-cocycle*. Similarly, the homogeneous elements of $\operatorname{im} d$ are called *coboundaries*, and a coboundary of degree p is called a p*-coboundary*.

We describe how every cosimplicial algebra (A, ∂, σ) gives rise to a differential graded algebra $\mathscr{D}A$. We set

$$\mathscr{D}A = \bigoplus A^p$$

and consider on $\mathscr{D}A$ the unique graded multiplication, which we call the *cup product*, that on homogeneous elements is as follows. If $a \in A^p$ and $b \in A^q$, we put

$$a \smile b = \partial^0 \cdots \partial^0(a) \cdot \partial^{m+n} \cdots \partial^{n+1}(b).$$

Note that this makes sense because $\partial^0 \cdots \partial^0(a)$ and $\partial^{m+n} \cdots \partial^{n+1}(b)$ are elements of the algebra A^{m+n}. The differential on $\mathscr{D}A$ is $d = \sum(-1)^i \partial^i$, the usual differential of the associated cochain complex of A. The cup product makes $(\mathscr{D}A, d)$ into a differential graded algebra, so that the cohomology of any simplicial algebra is canonically endowed itself with a graded algebra structure.

Let us show how this construction applies in our context. Let X be an $\mathbf{e}$-bicomodule which is a linearization of a species X_0 in Sp, and let $C^*(\mathbf{x})$ be the cosimplicial k-module we considered in Section 1. We now make it into a cosimplicial k-algebra.

Let q be a non-negative integer. The collection $C^q(\mathbf{x})$ of q-cochains is a k-algebra with product defined pointwise: if α, β are cochains, F is a decomposition of a finite set, we define the cochain $\alpha\cdot\beta$ with components

$$(\alpha\cdot\beta)(F)(z)=\alpha(F)(z)\cdot\beta(F)(z)$$

for every $z\in X_0(I)$, and extend this linearly to X. One can check that $\alpha\cdot\beta$ is indeed a morphism in Sp_k —in particular the required equivariance condition follows by checking it on basis elements in X_0— and that this makes $(C^*(\mathbf{x}),\partial,\sigma)$ into a cosimplicial algebra, that is, the coface and codegeneracies are algebra morphisms. As described above, $H^*(\mathbf{x})$ inherits a graded algebra structure from $C^*(\mathbf{x})$. We record the following for future reference, whose proof is a straightforward calculation:

PROPOSITION 3.1. *Let α be a p-cochain and let β be a q-cochain, both in $C^*(\mathbf{x})$. The cup product of α and β is the $(p+q)$-cochain with components*

$$(\alpha\smile\beta)(F)(z)=\alpha(F')(z\setminus\!\!\setminus F')\beta(F'')(z /\!\!/ F''). \tag{4}$$

for any decomposition $F=(F',F'')$ of a finite set I into $p+q$ blocks $F'=(F'_1,\dots,F'_p)$ and $F''=(F''_1,\dots,F''_q)$ and every structure $z\in X_0(I)$.

In Chapter V we briefly explain how one can, more generally, endow $C^*(\mathbf{x})$ with a cup product using a diagonal map $\Delta:\mathbf{x}\longrightarrow\mathbf{x}\square\mathbf{x}$.

With this at hand, we now compute some cohomology rings of species we defined in Chapter II, which are of interest to the authors of [**AM2010**]. Before doing so, we consider a useful operation on the algebra $C^*(\mathbf{x})$ that will aid us in such computations. This is in parallel with the ideas developed in Section 1.3.4 of [**Lod1998**].

4. Antisymmetrization of cochains

Fix an $\mathbf{e}$-bicomodule $\mathbf{x}$ in Sp_k. For each nonnegative integer q, the symmetric group S_q acts on decompositions of length q by permuting the blocks, so that for a permutation σ and a decomposition $F=(F_1,\dots,F_q)$, we have ${}^{\sigma}F=(F_{\sigma^{-1}(1)},\dots,F_{\sigma^{-1}(q)})$; it is important to remark this action is different from the action of S_I on $\mathbf{e}^{\otimes q}(I)$. Nonetheless, these two actions are evidently compatible.

We can turn this into an action of S_q on $C^q(\mathbf{x})$: for a cochain $\alpha:\mathbf{x}\longrightarrow\mathbf{e}^q$, we define another cochain α^{σ} such that for any composition $F=(F_1,\dots,F_q)$ of a finite set I and every structure $z\in X(I)$,

$$(\alpha^{\sigma})(F)(z)=\alpha({}^{\sigma}F)(z).$$

Extending this linearly, we obtain an action of the group algebra $k[S_q]$ on $C^q(\mathbf{x})$. We define the *antisymmetrization element* $\varepsilon_q \in k[S_q]$ to be

$$\varepsilon_q = \sum_{\sigma\in S_q} (-1)^\sigma \sigma.$$

We will think about ε_q both as an element of the group algebra of S_q and as an endomorphism $\varepsilon_q : C^q(\mathbf{x}) \longrightarrow C^q(\mathbf{x})$, and we omit the index q when it is not needed. The purpose of this section is to show that, because $\mathbf{e}$ is a cocommutative comonoid, we have $\varepsilon_p d = 0$ whenever $\mathbf{x}$ is a cosymmetric bicomodule. This sometimes allows to decide when a cocycle is not a coboundary: if α is a cocycle whose image under ε is nonvanishing, we can conclude α is not a coboundary.

LEMMA 4.1. *Let $q > 0$ be an integer, and fix $i \in \{1,\dots,q\}$.*

(1) If $\sigma \in S_{q+1}$ is such that $\sigma(q+1) = i$, let $\tau \in S_q$ be the permutation whose word is obtained from that of σ by replacing every letter j greater than i by $j-1$, then $(-1)^\tau(-1)^\sigma = (-1)^{q+1-i}$.

(2) If $\sigma \in S_{q+1}$ is such that $\sigma(1) = i$, let $\tau \in S_q$ be the permutation whose word is obtained from that of σ by replacing every letter j greater than i by $j-1$, then $(-1)^\tau(-1)^\sigma = (-1)^{i-1}$.

PROOF. We proceed as in Lemma 4.3. We may obtain the sign of a permutation σ by counting the number of inversions in σ. In the first part of the proposition, if we remove the letter i, which appears last in σ, we lose $q+1-i$ inversions corresponding to those j larger than i; in the second part, i appears first, and therefore we lose, in this case, $i-1$ inversions corresponding to those j smaller than i. ◀

With this at hand, we can prove the desired result, which we record in the form of the following proposition.

PROPOSITION 4.2. *Suppose X is a cosymmetric $\mathbf{e}$-bicomodule. Then $\varepsilon d = 0$ in $C^*(\mathbf{x})$. It follows that if α is a cocycle for which $\varepsilon\alpha \neq 0$, then α is not a coboundary.*

PROOF. We compute. If $q = 0$ then $d^0 = 0$, and the claim is obvious, so we may assume $q > 0$. Consider a q-cochain α, and fix a finite set I, a structure $z \in X(I)$ and a decomposition $(F_1,\dots,F_{q+1})$ of I. When computing $\varepsilon d\alpha(F)(z)$, the terms

$$\sum_{\sigma\in S_{q+1}} (-1)^\sigma \alpha(F_{\sigma(1)},\dots,F_{\sigma(i)} \cup F_{\sigma(i+1)},\dots,F_{\sigma(q+1)})$$

are all zero since $F_{\sigma(i)} \cup F_{\sigma(i+1)} = F_{\sigma(i+1)} \cup F_{\sigma(i)}$ while the associated permutations differ by a transposition, so the sum consists of terms that cancel in pairs. It remains to consider the sum

$$\sum_{\sigma\in S_{q+1}} (-1)^{\sigma}\alpha(F_{\sigma(2)},\dots,F_{\sigma(q+1)})(z \parallel F^c_{\sigma(1)}) \\ + (-1)^{\sigma+q+1}\alpha(F_{\sigma(1)},\dots,F_{\sigma(q)})(z \parallel F^c_{\sigma(q+1)}).$$

Grouping terms according to the value of $\sigma(1)$ and $\sigma(q+1)$, this becomes

$$\sum_{i=1}^{q+1}\sum_{\sigma(1)=i} (-1)^{\sigma}\alpha(F_{\sigma(2)},\dots,F_{\sigma(q+1)})(z \parallel F^c_{i}) \\ + \sum_{i=1}^{q+1}\sum_{\sigma(q+1)=i} (-1)^{\sigma+q+1}\alpha(F_{\sigma(1)},\dots,F_{\sigma(q)})(z \parallel F^c_{i}).$$

To conclude, we note these two sums cancel in view of the previous lemma. ◂

5. The cohomology ring of the species of linear orders

The aim of this section is to give a complete description of the cohomology ring of the species of linear orders endowed with its canonical cosymmetric bicomodule structure. This calculation, and the ones to follow in the next sections, were done before we developed the theory of Chapter IV. Although this theory provides an alternative —and much simpler!— form of computation, all what follows shows how this machinery was thought out and generalized.

Recall the species of linear orders L is such that, for each finite set I, $L(I)$ is the collection of linear orders $\ell = i_1\cdots i_t$ on I, and for each bijection $\sigma : I \longrightarrow J$ and each linear order $\ell = i^1\cdots i^t$ on I, $\sigma(\ell) = \sigma(i^1)\cdots\sigma(i^t)$. In other words, $L_n = S_n$ and the action of S_n on L_n is the regular left representation. The linearization kL of L has a canonical cosymmetric **e**-bicomodule structure obtained by restricting linear orders, and defines, by Theorem II.8.3, a cosimplicial object which we denote by $(C^*(L),\partial,\sigma)$, whose cohomology we want to compute.

For each non-negative integer p, let M_p be the free monoid on the letters $x_1,\dots,x_p$; in particular, M_0 is the free monoid on zero letters. Moreover, if for each $n \geqslant 0$ we consider the subset $M_p(n)$ consisting of those words of length n, we have that $M_p = \bigsqcup M_p(n)$. We can make the sequence of monoids $\{M_p\}_{p\geqslant 0}$ into a simplicial monoid

(M, d, s) as follows. Define homomorphisms $s_j : M_p \longrightarrow M_{p+1}$ and $d_i : M_{p+1} \longrightarrow M_p$ for $j \in \{1, \ldots, p-1\}$ so that on generators we have

$$d_0(x_i) = \begin{cases} 1 & \text{if } i = 1, \\ x_{i-1} & \text{if } 1 < i \leqslant p+1, \end{cases} \qquad d_j(x_i) = \begin{cases} x_{i-1} & \text{if } j < i \leqslant p+1, \\ x_i & \text{if } 1 \leqslant i \leqslant j, \end{cases}$$

$$d_p(x_i) = \begin{cases} x_i & \text{if } 1 \leqslant i \leqslant p, \\ 1 & \text{if } i = p+1, \end{cases} \qquad s_j(x_i) = \begin{cases} x_i & \text{if } i \leqslant j, \\ x_{i+1} & \text{if } i > j. \end{cases}$$

From this simplicial monoid we obtain a simplicial algebra $kM_\#$ by linearizing it degreewise, and in turn, taking duals, a cosimplicial k-module $kM^\# = \hom_k(kM_\#, k)$. Using this cosimplicial object, we will obtain another description of $(C^*(L), \partial, \sigma)$, and a first step toward this is the following lemma, whose proof we omit.

LEMMA 5.1. *There are inverse bijections $\varphi_q(n)$ and $\psi_q(n)$ between the set $F_q(n)$ of compositions of length q of $[n]$ and the set of words $M_q(n)$ on q letters $x_1, \ldots, x_q$ of length n, in such a way that:*

(1) A composition $F = (F_1, \ldots, F_q)$ is mapped by $\varphi_q(n)$ to the word $\omega_F = x_{i_1} \cdots x_{i_n}$, where $i_j = k$ if $j \in F_k$. In words, the letter in position j in ω_F has index equal to the block where j appears in F.

(2) A word $\omega = x_{i_1} \cdots x_{i_n}$ is mapped by $\psi_q(n)$ to the partition $F_\omega = (F_1, \ldots, F_q)$ where $F_k = \{j \in [n] : i_j = k\}$. In words, F_k consists of those positions where letter k appears in ω.

(3) Moreover, a word in $M_q(n)$ is sent to a composition *in $F_q(n)$ if and only if every letter appears in it. We call such words* sincere. ◀

With this at hand, we can obtain the desired alternative description of $(C^*(L), \partial, \sigma)$.

PROPOSITION 5.2. *There is an isomorphism of cosimplicial k-modules $kM^\# \to C^*(L)$.*

PROOF. We construct inverse bijections $\varphi_p : kM^p \longrightarrow C^p(L), \psi_p : C^p(L) \longrightarrow kM^p$ that constitute morphisms of cosimplicial k-modules.

Consider a p-cochain $\alpha : kL \longrightarrow E^p$. For each integer $n \geqslant 0$, $kL([n])$ is a free S_n-module of rank one, and from this it follows α is determined uniquely on the set $[n]$ by the values $\alpha(F_1, \ldots, F_p)(\mathrm{id}_n)$ where id_n is the usual linear order of $[n]$, and $F = (F_1, \ldots, F_p)$ is a decomposition of $[n]$. Indeed, let F' be a decomposition of $[n]$ into p blocks, and let ℓ be a linear order on $[n]$. There is a unique bijection σ that transports

ℓ to id_n, and then

$$\alpha(F_1,\ldots,F_p)(\ell) = \alpha(F_1',\ldots,F_p')(\mathrm{id}_n).$$

Using the bijections of the lemma, we define a function $\varphi^p(\alpha) : M_p \longrightarrow k$ so that for each $\omega \in M_p$ of length n we have $\varphi(\alpha)(\omega) = \alpha(F_\omega)(\mathrm{id}_n)$. Conversely, a function $f : M_p \longrightarrow k$ defines a p-cochain $\psi^p(f) : L \longrightarrow E^{\otimes p}$ by setting $\psi(f)(F)(\mathrm{id}_n) = f(\omega_F)$ whenever $F \in F_p(n)$. Because $\varphi_p(n)$ and $\psi_p(n)$ define inverse bijections from $M_p(n)$ to $F_p(n)$, and because we already noted a p-cochain is determined uniquely by the values $\alpha(F)(\mathrm{id}_n)$ for $n \geqslant 0$ and $F \vdash [n]$, it follows that φ^p and ψ^p are, for each non-negative integer p, inverse bijections from kM^p to $C^p(L)$.

It remains to show $\varphi = (\varphi^p)$ and $\psi = (\psi^p)$ are morphisms of cosimplicial k-modules, that is, we must check that for all non-negative p and $i \in \{0,\ldots,p+1\}$ the following equalities hold:

$$\varphi^p d^i \psi^p = \partial^i, \qquad \varphi^{p+1} s^i \psi^{p+1} = \sigma^i. \tag{5}$$

Fix a p-cochain $\alpha : kL \longrightarrow E^{\otimes p}$. If $j \in \{0,\ldots,p+1\}$, we already observed that

$$\sigma^j(\alpha)(F_1,\ldots,F_p)(z) = \alpha(F_1,\ldots,F_{j-1},\varnothing,F_j,\ldots,F_p)(z),$$

which makes it clear the second equality in (5) holds, since, under our bijection, the appearance of the empty block has the effect of shifting the letters x_i for $i \geqslant j$ up by one in their subindices. Similarly, for $i \in \{1,\ldots,p\}$, we have

$$\partial^i(\alpha)(F_1,\ldots,F_{p+1})(z) = \alpha(F_1,\ldots,F_i \cup F_{i+1},\ldots,F_{p+1})(z)$$

so the first equality in (5) holds for such values of i, by an analogous argument as the one we just carried out for the codegeneracy maps. It remains to deal with the cases where i equals either 0 or $p+1$, in which we have

$$\begin{aligned}\partial^0(\alpha)(F_1,\ldots,F_{p+1})(z) &= \alpha(F_2,\ldots,F_{p+1})(z \parallel F_1^c),\\ \partial^{p+1}(\alpha)(F_1,\ldots,F_{p+1})(z) &= \alpha(F_1,\ldots,F_p)(z \parallel F_{p+1}^c).\end{aligned}$$

Let $(F_1,\ldots,F_{p+1})$ be a decomposition of $[n]$, so that $(F_2,\ldots,F_{p+1})$ is a partition of $T = [n] \setminus F_1$. Let ℓ be the order on T induced by id_n. The bijection $\ell_j \longmapsto j$ sends ℓ to id_k if T has k elements, the composition $(F_2,\ldots,F_{p+1})$ to some other composition

$(F'_2,\dots,F'_{p+1})$ of $[k]$, and naturality of α guarantees that

$$\alpha(F_2,\dots,F_{p+1})(\ell) = \alpha(F'_2,\dots,F'_{p+1})(\mathrm{id}_k).$$

This makes it evident the first equality of (5) holds when $i = 0$, since the words assigned to $(F'_2,\dots,F'_{p+1})$ is obtained from that of $(F_1,F_2,\dots,F_{p+1})$ by deleting the first letter x_1 and re-indexing the remaining words accordingly; an analogous observation holds for the last differential. ◀

We can illustrate the last argument of the proof: if F is the composition $(F_1,F_2,F_3) = (\{1,5\},\{2,4\},\{3,6,7\})$ of $[7]$ then $T = \{2,3,4,6,7\}$ and

$$\alpha(F_2,F_3)(23467) = \alpha(\{1,3\},\{2,4,5\})(12345).$$

The word assigned to F is $\omega_F = x_1x_2x_3x_2x_1x_3x_3$, while the word assigned to the composition $(\{1,3\},\{2,4,5\})$ is $x_1x_2x_1x_2x_2$, which is obtained from ω_F by setting $x_1 = 1$ and shifting the remaining indices, as per our definition of $d_0(\omega_F)$.

Assume now k is a PID, and let us compute the homology of the simplicial k-module $kM_\#$. To do so, we will use a spectral sequence coming from the filtration on $kM_\#$ by word-length. Once we check this has k-free homology, we will be able to deduce, by an appeal of the Universal Coefficient Theorem, that the cohomology of $kM^\#$ is dual to this.

To compute $H_*(kM_\#)$ we will use the normalized complex $k\overline{M}$ associated to kM. The description of degeneracy maps of M makes it clear that $k\overline{M}_p$ has basis the sincere words on the letters $x_1,\dots,x_p$, and, as usual, the differential is induced from that of kM. There is a filtration on the complex $k\overline{M}$ that has $F_pk\overline{M}_q = \bigoplus_{n\leqslant p}\overline{M}_q(n)$. This filtration is exhaustive and bounded below, so the spectral sequence associated to it converges to $H_*(kM_\#)$ by Theorem A.2.1 in the Appendix. The zero page of this spectral sequence has

$$E^0_{p,q} = F_pk\overline{M}_{p+q}/F_{p-1}k\overline{M}_{p+q},$$

a free k-module with basis the sincere words of length p on $p+q$ letters. Observe that $E^0_{p,q}$ vanishes when $p+q \leqslant 0$ and $p \neq 0$, when $q > 0$ or when $p < 0$. In the first case, this is obvious when $p+q < 0$, and when $p+q = 0$ the only nontrivial filtration quotient is $E^0_{0,0}$; in the second case, this is because there are no sincere words of length at most p

in $p+q>p$ letters, while the last case follows from the fact that $F_p = 0$ for $p<0$. Our spectral sequence is therefore concentrated in a cone as illustrated in Figure 1.

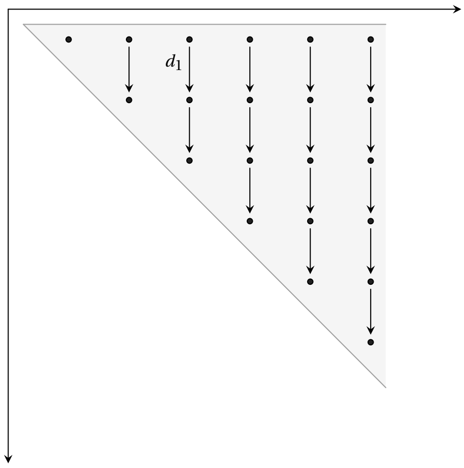


FIGURE 1. The spectral sequence for L lies in a cone in the fourth quadrant.

To begin the computation of the E^1-page, we note the differentials d_0 and d_{p+q} strictly decrease word length, so they vanish modulo the filtration. It follows the differential $d^0 : E^0_{p,q} \longrightarrow E^0_{p,q-1}$ is the alternating sum of the inner coface maps of M. To continue, we must compute the homology of the columns in E^0, and to do so we identify the complexes that appear as such.

DEFINITION 5.3. Let X be a non-empty finite set. For each integer $j \geqslant -1$, write $C_j(\Sigma, X)$ for the free k-module with basis the compositions of X of length $j+2$. There are face maps $\partial_i : C_j(\Sigma, X) \longrightarrow C_{j-1}(\Sigma, X)$ such that

$$\partial_i(F_0,\ldots,F_{j+1}) = (F_0,\ldots,F_i \cup F_{i+1},\ldots,F_{j+1})$$

that make $C_*(\Sigma, X)$ into a semisimplicial k-module.

We will show in Chapter IV, Section 2, that $C_*(\Sigma, X)$ has homology equal to the reduced homology of an $(r-2)$-sphere with coefficients in k, with r the cardinal of X, and that the generator of $H_{r-2}(C_*(\Sigma, X))$ is the element

$$\nu_r = \sum_{\sigma\in S_r} (-1)^\sigma \sigma,$$

where a composition of X into r blocks is identified with a permutation of X via a choice of linear ordering of X[2]. We exemplify this for $X = [3]$ in Figure 2. The resulting complex $C_*(\Sigma, X)$ is, in fact, that of a triangulation of S^1 induced from the braid arrangement in $\mathbb{R}^3$.

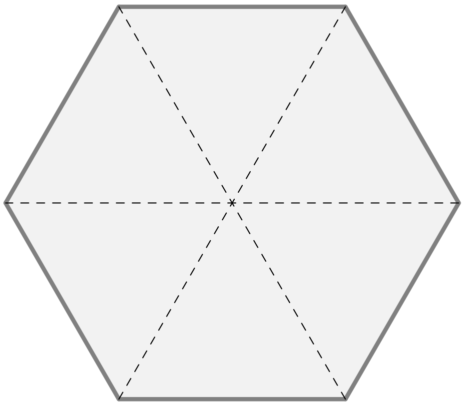


FIGURE 2. The complex $C_*(\Sigma, [3])$ is the simplicial complex of a triangulation of S^1. The the braid arrangement is drawn with dotted lines, the triangulation of D^2 is light gray, and the triangulation of S^1 is marked with heavy gray lines.

With this information at hand, we can identify the E^0-page of this spectral sequence.

PROPOSITION 5.4. *For each integer $p \geqslant 1$, the complex $E^0_{p,*}$ identifies with the complex $C_{p-*}(\Sigma, [p])[2]$, that is, for each $q \geqslant 0$ we have identifications $E^0_{p,-q} = C_{p+q-2}(\Sigma, [p])$. It follows that the E^1-page of the spectral sequence is concentrated in the positive p-axis. Moreover, under this identification, the differential $d^1 : E^1_{*,0} \longrightarrow E^1_{*-1,0}$ is identically zero.*

PROOF. We already know that a sincere word of length p in $p - r$ letters corresponds uniquely to a composition of $[p] = \{1, \ldots, p\}$ into $p - r$ blocks. Thus, at least as k-modules, we have the desired identifications $E^0_{p,-q} = C_{p+q-2}(\Sigma, [p])$ for each $q \geqslant 0$. It is easy to verify this identification is compatible with the face maps. Our first claim now follows from the remarks preceding the proposition. The last claim, on the other hand, is immediate, since the generator we chose for $E^1_{*,0}$ has zero differential in our original complex and not only modulo the filtration. ◀

We can now conclude the determination of $H^*(L)$ as a graded k-module.

[2]This can be done coherently if we assume that X is a subset of $\mathbb{N}$, which we usually do.

COROLLARY 5.5. *For each integer $q \geqslant 0$, the k-module $H_q(kM_\#)$ is free of rank one. It follows that $H^q(kM_\#)$ is k-free of rank one for each integer $q \geqslant 0$, and then the same is true for $H^q(L)$.*

Having identified the ranks of the cohomology groups of L, we now give explicit generating cocycles and the algebra structure of $H^*(L)$. Recall we defined in Section 2 a 2-cocycle sch in $C^2(L)$, which we call the Schubert cocycle. Along with this, we have a 1-cocycle κ in $C^1(L)$, the cardinality cocycle. Our final result is the following.

THEOREM III.5.6. *Let k be a PID as before, and suppose additionally that $\mathbb{Q}$ is contained in the field of fractions of k. The cohomology algebra $H^*(L)$ is generated by the cardinality and the Schubert cocycles, and there is an isomorphism of graded algebras*

$$H^*(L) \simeq k[X] \otimes k[Y]/(Y^2)$$

where $|X| = 2$ and $|Y| = 1$.

To prove this, we will show that sch and κ generate $H^*(L)$ as an algebra, and we will do this with the ideas of Section 4. We will denote by sch' the 2-cocycle $\kappa \smile \kappa - \mathrm{sch}$, which is such that

$$\mathrm{sch}'(S,T)(\ell) = \#\{(i,j) \in S \times T : i < j \text{ according to } \ell\}.$$

It is not hard to see that $2\kappa \smile \kappa = -d\sigma$ where $\sigma(I)(\ell) = |I|^2$, so the cohomology classes of sch and $-\mathrm{sch}'$ are the same, and then it suffices we prove the following.

LEMMA 5.7. *With the hypotheses of the theorem, for each non-negative integer p, the cocycles $\kappa \smile \mathrm{sch}'^p$ and sch'^p are not coboundaries.*

PROOF. Let ε be the antisymmetrization map. In view of Lemma 4.2, it suffices we prove that for every non-negative integer p we have,

$$\varepsilon(\mathrm{sch}'^p)(1,2,\ldots,2p)(\mathrm{id}) = p!, \qquad \varepsilon(\kappa \smile \mathrm{sch}'^p)(1,2,\ldots,2p+1)(\mathrm{id}) = p!. \tag{6}$$

It follows that $\kappa \smile \mathrm{sch}'^p$ and sch'^p are not coboundaries as long as $p!$ is invertible in the field of fractions of k, as in our case. To prove the first equality we note that, by definition,

$$\varepsilon(\mathrm{sch}'^p)(1,2,\ldots,2p)(\mathrm{id}) = \sum_{\sigma \in S_{2p}} (-1)^\sigma \mathrm{sch}'(\sigma(1)\sigma(2)) \cdots \mathrm{sch}'(\sigma(2p-1)\sigma(2p))$$

where $\mathrm{sch}'(ij)$ is 1 if $i<j$ and zero if not. It follows that in the right hand side the only terms that do not vanish are those where $\sigma(1)<\sigma(2),\dots,\sigma(2p-1)<\sigma(2p)$. Let T_p be the collection of permutations in S_{2p} such that

$$\sigma(1)<\sigma(2),\dots,\sigma(2p-1)<\sigma(2p),$$

$$\sigma(1)<\sigma(3)<\cdots<\sigma(2p-1),$$

and let us show our sum equals

$$p!\sum_{\sigma\in T_p}(-1)^{\sigma}.$$

Indeed, interchanging two factors $\mathrm{sch}'(ij)$ and $\mathrm{sch}'(kl)$ in a term of our sum amounts to a double transposition, and this does not affect the sign of a permutation. If follows that to each $\sigma\in T_p$ there are associated $p!$ terms, all with sign $(-1)^{\sigma}$.

Our claim is then equivalent to the equality $\sum_{\sigma\in T_p}(-1)^{\sigma}=1$. To prove the latter, we define an involution $u:T_p\longrightarrow T_p$ so that $u(\mathrm{id})=\mathrm{id}$ and such that for every other permutation we have $(-1)^{u(\sigma)}=-(-1)^{\sigma}$. We may look at a permutation $\sigma\in T_p$ as a

$\sigma(1)$	$\sigma(2)$
$\sigma(3)$	$\sigma(4)$
$\vdots$	$\vdots$
$*$	$\sigma(2p)$

FIGURE 3. The tableau associated to $\sigma\in T_p$.

$p\times 2$ tableau T_σ with increasing rows and increasing first column as in Figure 3. If T_σ is not the identity permutation then there is a first row of the form

i	$i+k$

with $k>1$ — call such pair an exceedance of T_σ. The row immediately below is of the form

$i+1$	$i+j$

with $j > 1$: by our condition on rows and columns, $i+1$ must appear immediately under i in T_σ. If we transpose $i+k$ and $i+j$, we obtain a valid tableau $u(T_\sigma)$ in T_p, as illustrated in Figure 4. Moreover, the row at which the first exceedance of T_σ

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \vdots & \vdots \\ \hline i & i+k \\ \hline i+1 & i+j \\ \hline \vdots & \vdots \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \vdots & \vdots \\ \hline i & i+j \\ \hline i+1 & i+k \\ \hline \vdots & \vdots \\ \hline \end{array}$$

FIGURE 4. The involution $u: T_p \longrightarrow T_p$.

happens is preserved so that u is involutive, and the sign of the permutation associated to $u(T_\sigma)$ is $-(-1)^\sigma$: T_σ and $u(T_\sigma)$ differ by a transposition. This proves that $\sum_{\sigma\in T_p}(-1)^\sigma = 1$, as desired.

The second equality in (6) follows from the first one. Indeed, by definition

$$\begin{aligned} &\varepsilon(\kappa \smile \mathrm{sch}'^{p})(1,2,\ldots,2p+1)(\mathrm{id}) \\ &\qquad = \sum_{\sigma\in S_{2p+1}} (-1)^\sigma \mathrm{sch}'(\sigma(2)\sigma(3))\cdots\mathrm{sch}'(\sigma(2p)\sigma(2p+1)), \end{aligned}$$

and grouping the terms of this sum according to the value of $\sigma(1)$, we can write it in the form

$$\sum_{i=1}^{2p+1} \sum_{\sigma(1)=i} (-1)^\sigma \mathrm{sch}'(\sigma(2)\sigma(3))\cdots\mathrm{sch}'(\sigma(2p)\sigma(2p+1)),$$

and for each i, the corresponding inner sum equals

$$(-1)^{i-1} \sum_{\sigma\in S_{2p}} (-1)^\sigma \mathrm{sch}'(\sigma(1)\sigma(2))\cdots\mathrm{sch}'(\sigma(2p-1)\sigma(2p))$$

by virtue of Lemma 4.3. We conclude that the original sum is exactly $p!$, since we already calculated this last sum. ◀

OBSERVATION 5.8. It is easy to see that $\#T_p = (2p-1)!!$, since to each element of T_p there correspond, as in our proof, $2^p p!$ elements of S_{2p}. This gives the sum must be nonzero modulo two, for it is a sum of an odd number of signs. This is faster, but less precise than the formulas of the proof, and it might be of interest to consider the cases when k is a finite field of positive characteristic.

6. The cohomology ring of the species of compositions

We proceed to calcuate the cohomology of Σ. The method is analogous to what was done in the previous section, and we will in fact exploit the determination of $H^*(L)$ to determine $H^*(\Sigma)$ using the morphism $L \longrightarrow \Sigma$. To begin, the following will organise our calculations.

DEFINITION 6.1. Fix a finite set I and two decompositions F, G of I. We define the *weight matrix* of the ordered pair (F, G) to be the matrix $\omega(F, G)$ such that $\omega(F, G)_{ij} = \#(F_i \cap G_j)$. Observe that the rows (respectively columns) of $\omega(F, G)$ are nonzero precisely when F (respectively G) is a composition of I.

The following is immediate.

LEMMA 6.2. *Let I be a finite set and let J be a finite set. Fix a pair of decompositions (F, G) of I and (F', G') of J. There exists a bijection $\sigma : I \longrightarrow J$ such that $(\sigma F, \sigma G) = (F', G')$ if and only if the weight matrices $\omega(F, G)$ and $\omega(F', G')$ coincide.*

Observe that any $p \times q$ matrix with entries in $\mathbb{N}_0$ arises in this way as a weight matrix. Indeed, if we pick any $p \times q$ matrix (A_{ij}) with non-negative integer entries, we can consider a $p \times q$ square grid, and start filling each square with the numbers from 1 to $n = \sum A_{ij}$ with the specified number of elements in each square. This produces a finite set I of n elements and decompositions (F, G) of I with $\omega(F, G) = n$.

Consider now a decomposition F of a finite set I into q blocks $(F_1, \ldots, F_q)$ and another decomposition G of I. Viewing the matrix $\omega(F, G)$ row-wise, we identify it with a tuple of vectors $(v_1 | \cdots | v_q) \in (\mathbb{N}^j)^q$ where j is the length of G. In particular, if $j = 0$, then I is the empty set.

DEFINITION 6.3. Fix a non-negative integer q. For each $j \geqslant 1$, we write $T_q(j)$ for the set of tuples $(v_1, \ldots, v_q) \in (\mathbb{N}^j)^q$ such that its associated matrix —obtained by viewing the v_i as column vectors— has nonzero rows. For $0 < i < q$, there are face maps $\partial_i : T_q(j) \longrightarrow T_{q-1}(j)$ so that

$$\partial_i(v_1, \ldots, v_q) = (v_1, \ldots, v_i + v_{i+1}, \ldots, v_q).$$

We define edge maps $\partial_0, \partial_q : T_q(j) \longrightarrow T_{q-1}(j) \cup T_{q-1}(j-1) \cup \cdots$ that delete the first and last vector of $(v_1, \ldots, v_q)$. If the matrix represented by the image of ∂_0 or ∂_q contains a zero rows, we delete it, to obtain an element of $T_{q-1}(j-1)$. By convention, each $T_q(0)$ is free of rank one. There are also degeneracy maps that add a column of zeros.

The above defines a simplicial k-module (C_*, ∂, σ) where $C_q = \bigoplus_{j\geqslant 0} T_q(j)$, and we can picture this is the following way, to obtain a complex (C_*, d):

$$
\begin{tikzcd}
C_q: \arrow[d] & T_q(0) \arrow[d, bend left] & T_q(1) \arrow[d, bend left] \arrow[dl, dotted] & T_q(2) \arrow[d, bend left] \arrow[dl, dotted] & T_q(3) \arrow[d, bend left] \arrow[dl, dotted] & \cdots \arrow[dl, dotted] \\
C_{q-1}: & T_{q-1}(0) \arrow[u, bend left] & T_{q-1}(1) \arrow[u, bend left] & T_{q-1}(2) \arrow[u, bend left] & T_{q-1}(3) \arrow[u, bend left] & \cdots
\end{tikzcd}
$$

Upon normalization, we can restrict ourselves to the spaces $\overline{T}_q(j)$ of $j \times q$ matrices with non-negative integer coefficients and both nonzero rows and nonzero columns. We define the weight of a matrix with non-negative integer coefficients as the sum of its entries, and write $\omega(A)$ for such number.

PROPOSITION 6.4. *There is an isomorphism of cosimplicial k-modules $C^*(\Sigma) \to C^*$ where C^* denotes the k-dual of C_* defined above.*

PROOF. Consider a p-cochain $\alpha : \Sigma \longrightarrow E^{\otimes p}$. Then α is determined uniquely by the collection of scalars $\{\alpha(F)(G) : F \vdash [n], G \models [n]\}$, where $G \models [n]$ denotes a composition of $[n]$. We claim that the scalar $\alpha(F)(G)$ depends only on the weight matrix $\omega(F,G)$. This is clear, because the weight matrix depends uniquely on the S_n-orbit of the pair (F,G) by the first lemma of this section. It follows that α determines uniquely a map $\Gamma(\alpha) : C_p \longrightarrow k$ that assigns a matrix of the form $\omega(F,G)$ in C_p the scalar $\alpha(F)(G)$. Conversely, every map $\theta : C_p \longrightarrow k$ defines a p-cochain $\Psi(\theta) : \Sigma \longrightarrow E^{\otimes p}$ such that $\alpha(F)(G) = \theta(\omega(F,G))$. A straightforward but tedious calculation shows Γ and Ψ are inverse cosimplicial morphisms, so the claim follows. ◀

Again, we assume k is a PID. Our objective is to determine the homology groups of C_*, and use the Universal Coefficient Theorem to determine those of C^*.

We already noted we can view the component C_q as the collection of matrices with q columns and a finite number of nonzero rows. The weight of such a matrix is the sum of its entries, and by construction the coface maps ∂_i for $0 < i < q$ preserve this weight, while ∂_0, ∂_q strictly decrease it. It follows there is a filtration of $\overline{C}_*$ by the subcomplexes consisting of those matrices with weight at most p, which we denote by F_pC_q. This filtration is manifestly bounded below and exhaustive, and provides with a convergent spectral sequence starting at $E^0_{p,q}$, a free k-module with basis the matrices with nonzero $p+q$ columns, nonzero rows and with weight exactly p. By the remarks on the coface maps, the differential on E^0 is the alternating sum of the inner coface maps of C_*.

If A is a matrix in $E^0_{p,q}$, denote by $\omega^r(A)$ the vector obtained by summing the *rows* of A, that is, $\omega^r(A)_j = \sum A_{ij}$. Call this the weight row vector of A. Note then that our differential $d^0 : E^0 \longrightarrow E^0$ preserves such sum, and it also preserves the number of rows of a matrix. It follows we have a direct sum decomposition

$$E^0_{p,*} = \bigoplus_{j\leqslant p} E^0_{p,*}(j),$$

and in turn a direct sum decomposition

$$E^0_{p,*}(j) = \bigoplus_{\omega\in\mathbb{N}^j} E^0_{p,*}(\omega),$$

where $E^0_{p,*}(j)$ is the subcomplex of $E^0_{p,*}$ consisting of those matrices with exactly j rows, and $E^0_{p,*}(\omega)$ is the subcomplex of $E^0_{p,*}(j)$ consisting of those matrices with j rows and row weight vector ω. A simple extension of the definition of the complexes $C_*(\Sigma, X)$ allows us to identify the complexes $E^0_{p,*}(\omega)$, like we did with the species of linear orders.

DEFINITION 6.5. Let M be a finite multiset on a set X, that is, a function $f : X \longrightarrow \mathbb{N}_0$, so that the collection of multisets on X is a monoid under addition. We say M is *proper* if $f(X) \subseteq \{0,1\}$. A *decomposition of* M is an ordered tuple $F = (f_1,\ldots,f_r)$ of multisets on X whose sum is f, and a *composition of* M is a decomposition with nonzero entries. Denote by $C_*(\Sigma, M)$ the graded vector space with basis the compositions of M, graded by composition length. This is a chain complex $C_*(\Sigma, M)$ with differential

$$\partial(f_0,\ldots,f_p) = \sum_{i=1}^{p-1} (-1)^i (f_1,\ldots,f_i + f_{i+1},\ldots,f_p).$$

Note that if M is the multiset on X that assigns every $x \in X$ the value 1, then $C_*(\Sigma, M)$ is the complex of Definition 5.3. One can also see, for example, that the complex associated to the multiset $\{11\cdots 11\}$ consisting of $d > 1$ ones arises from the canonical triangulation of the $(d-2)$-simplex. We have the following proposition, that completes the determination of the homology of the complexes $C_*(\Sigma, M)$:

PROPOSITION 6.6. *The complexes $C^*(\Sigma, M)$ are acyclic if M is a multiset that is not proper. In fact, if M is not proper and has d elements counted with multiplicity, the chain complex $C_*(\Sigma, M)$ arises as the simplicial chain complex associated to a triangulation of the $(d-2)$-disk induced from the braid arrangement.*

PROOF. We provide a sketch of the proof. Let M be a multiset with d elements, and associate to it the set $[d]$, without loss of generality, assume the multiset is obtained by identifying certain elements of d into blocks, say:

$$i_1^{(1)} < \cdots < i_{j_1}^{(1)} \mid \cdots \mid i_1^{(t)} < \cdots < i_{j_t}^{(t)}.$$

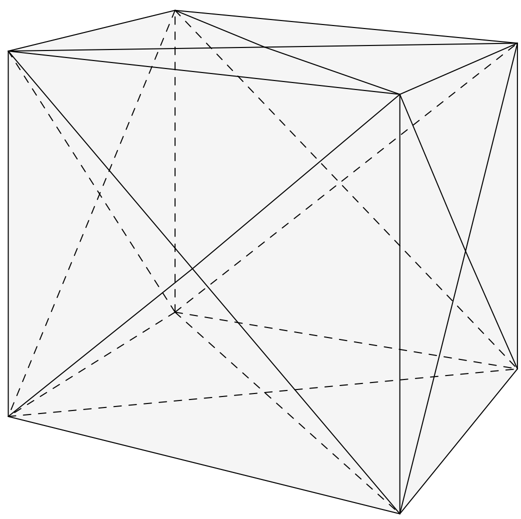


FIGURE 5. The triangulation of S^2 arising from the braid arrangement with corresponding complex $C_*(\Sigma, [4])$.

We can assume there is at least one block with more than one element, since the case X is a set has already been addressed. We refer the reader to the proof of Proposition 2.1 for the definition of the triangulation K of S^{d-2} arising from the braid arrangement. The simplices of K are in bijection with the compositions of $[d]$. We now consider the positive half-spaces of the hyperplanes $x_s = x_r$ whenever s and r are in the same block of the multiset M. The intersection of S^{d-2} with such halfspaces is a $(d-2)$-disk, and the intersection L of K with such halfspaces is a subcomplex, and in this way we obtain a bijection between compositions of M and simplices of L, in such a way that the complex that computes the simplicial homology of L is exactly $C_*(\Sigma, M)$. This completes the sketch of the proof. ◀

To illustrate, the complex $C_*(\Sigma, 1123)$ arises from the triangulation of the 2-disk obtained by slicing in half the triangulation of S^2 as shown in Figure 6, while the complex $C_*(\Sigma, 1122)$ arises from the triangulation of D^2 as shown in Figure 7, which can be obtained by further dividing the triangulation for 1123 by a second half-space. We now associate to a vector $\omega \in \mathbb{N}^j$ of weight p the p-multiset $M(\omega) = \{1^{\omega_1}, \ldots, j^{\omega_j}\}$, and obtain the following.

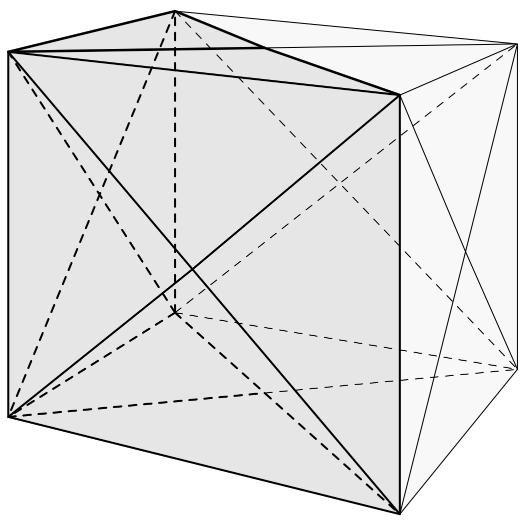


FIGURE 6. Shaded in black is the subcomplex that triangulates D^2 and corresponds to the multiset 1123.

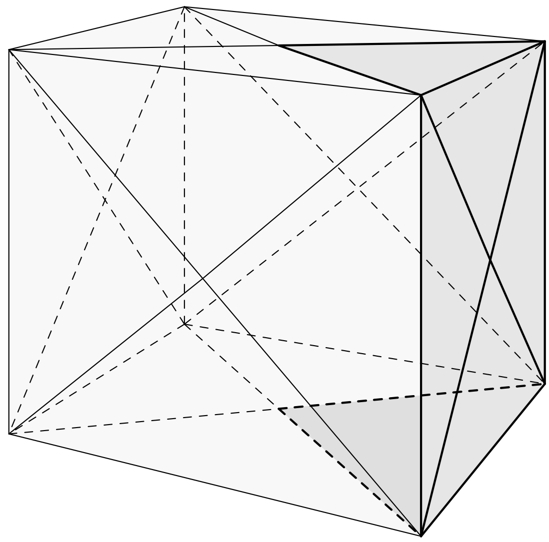


FIGURE 7. Shaded in black is the triangulation of D^2 arising from the multiset 1122.

PROPOSITION 6.7. *There is an isomorphism of complexes* $\Phi : E^0_{p,*}(\omega) \longrightarrow C_*(\Sigma, M(\omega))$ *that assigns to a* $s \times j$ *matrix* Z *the composition* F *of length* s *of* $M(\omega)$ *so that* $F_k = \{1^{Z_{k1}}, \dots, j^{Z_{kj}}\}$.

PROOF. This is now a straightforward verification. ◀

Because we have completely described the homology of the complexes $C_*(\Sigma, ?)$, we obtain the following corollary.

COROLLARY 6.8. *The* E^1 *page of the spectral sequence is concentrated in the* p*-axis, where* $E^1_{p,0}$ *is* k*-free of rank one for every non-negative integer* p*. Moreover* $d^1 = 0$*, so that* $H^p(\Sigma) = k$ *for every non-negative integer* p*.*

PROOF. The only vector ω whose associated multiset is proper is $e = (1, \dots, 1)$, so the only summand of $E^0_{p,*}$ with nontrivial homology groups is $E^0_{p,*}(e)$, and this has the reduced homology of a sphere. As in the case of linear orders, one checks the generator of $E^1_{p,0}$ has zero differential before looking at its class, so the spectral sequence degenerates where stated. ◀

We can now exhibit generators of $H^1(\Sigma)$ and $H^2(\Sigma)$, and deduce that the inclusion $L \longrightarrow \Sigma$ induces a quasi-isomorphism $C^*(\Sigma) \longrightarrow C^*(L)$.

DEFINITION 6.9. The *Schubert cochain* $\mathrm{sch} : \Sigma \longrightarrow E^{\otimes 2}$ is such that for a finite set I, a decomposition (S, T) of I and any composition F of I,

$$\mathrm{sch}(S,T)(F) = \#\{(i,j) \in I \times J : i > j \text{ according to the blocks of } F\}.$$

We also recall that the cardinality cocycle $\kappa : \Sigma \longrightarrow E$ is such that $\kappa(I)(F) = \#I$. With this at hand, our result is the following.

PROPOSITION 6.10. *The map $\iota^* : C^*(\Sigma) \longrightarrow C^*(L)$ induced by the inclusion $L \longrightarrow \Sigma$ is a quasi-isomorphism. Moreover, $H^*(\Sigma)$ is generated as an algebra by the classes of the Schubert cocycle and the cardinality cocycle.*

PROOF. This is now a straightforward verification. It is immediate that the arrow ι^* maps the Schubert cocycle of Σ to the Schubert cocycle of L, and the cardinality cocycle of Σ to that of L. Because we already checked the latter have non-trivial classes in $H^*(L)$, the same is true for the former in $H^*(\Sigma)$. Moreover, we have checked that for each non-negative integer q, the k-module $H^q(\Sigma)$ is free of rank one, so it is generated, like $H^q(L)$, by $\mathrm{sch}^{q/2}$ or $\kappa \smile \mathrm{sch}^{(q-1)/2}$ according to whether q is even or odd. ◀

7. Determined and representable species

The special case of posets. For Λ a finite poset define the set species $\mathbf{x}_\Lambda$ that sends a finite set I to the collection $\mathbf{x}_\Lambda(I)$ of order morphisms $f : \Lambda \longrightarrow \wp(I)$, and sends a bijection $\sigma : I \longrightarrow J$ to the map $\mathbf{x}_\Lambda(\sigma) = \wp(\sigma) \circ f : \Lambda \longrightarrow \wp(J)$. If $h : \Lambda_1 \longrightarrow \Lambda_2$ is an order morphism between posets, define a species morphism $h^* : \mathbf{x}_{\Lambda_2} \longrightarrow \mathbf{x}_{\Lambda_1}$ that acts by precomposition. This gives a functor between the category of posets P and Sp.

If there is defined a set comodule structure on the species of parts, we can define a set comodule structure on $\mathbf{x}_\Lambda$ as follows: for a set I, a subset S and a map $f : \wp(I) \longrightarrow \Lambda$, let $f \parallel S : \Lambda \longrightarrow \wp(S)$ be the map that sends $\lambda \in \Lambda$ to $f(\lambda) \parallel S$. Of course, one can also define a bicomodule structure if such a structure is defined on the species of parts. In what follows we will always assume $\mathbf{x}_\Lambda$ is endowed with the cosymmetric bicomodule structure obtained from the canonical structure defined on the species of parts.

If Λ is a finite poset, an *order ideal* of Λ is a subset $J \subseteq \Lambda$ such that whenever $s \in J$ and $t \leqslant s$, it follows that $t \in J$. An *antichain* in Λ is a subset $A \subseteq \Lambda$ of incomparable elements. There is a bijection between antichains and order ideals in such a way that an order ideal J is sent to the collection $A(J)$ of maximal elements of J, and an antichain A is sent to the order ideal $J(A) = \{s \in \Lambda : s \leqslant t \text{ for some } t \in A\}$. The set of order ideals of a poset Λ, denoted by $J(\Lambda)$, is itself a poset under the order of set containment. We begin with a lemma that shows how order ideals are related to $H^1(\mathbf{x}_\Lambda)$.

LEMMA 7.1. *Let Λ be a finite poset and $J(\Lambda)$ the set of order ideals of Λ.*

(1) For each $J \in J(\Lambda)$ there is defined a 1-cocycle $\alpha_J : \mathbf{x}_\Lambda \longrightarrow E$ such that for every finite set I and every order morphism $z : \Lambda \longrightarrow \wp(I)$,

$$\alpha_J(I)(z) = \#\{i \in I : (\forall \lambda \in \Lambda)(i \in z(\lambda) \iff \lambda \notin J)\}$$

(2) The set of cocycles $\{\alpha_J : J \in J(\Lambda)\}$ forms a basis for the k-module of 1-cocycles.

PROOF. In words, $\alpha_J(I)(z)$ is the number of elements in I that are in none of the sets $z(\omega)$ for $\omega \in J$, and are in every set $z(\omega)$ for $\omega \notin J$. The definition of α_J makes it evident that it is a cocycle, and proves the first claim.

Consider now an arbitrary 1-cocycle $\alpha : \mathbf{x}_\Lambda \longrightarrow E$ so that for any decomposition (S, T) of a finite set I and any $z \in \mathbf{x}_\Lambda(I)$,

$$\alpha(I)(z) = \alpha(S)(z \parallel S) + \alpha(T)(z \parallel T).$$

Define a function $\varphi(\alpha) : \mathbf{x}_\Lambda([1]) \longrightarrow k$ that sends $z \mapsto \alpha(\{1\})(z)$. The claim is that α is uniquely determined by $\varphi(\alpha)$, as we stated in Proposition 1.3. Indeed, consider a finite set I with n elements and fix a bijection $\iota : [n] \longrightarrow I$. This defines a decomposition $(F_1, \ldots, F_n)$ of I where $F_j = \{\iota(j)\}$, and there is a unique bijection $f_j : F_j \longrightarrow [1]$. Using the cocycle equation several times, we find that

$$\alpha(I)(z) = \sum_{j=1}^{n} \alpha(F_j)(z \parallel S_j)$$

and because α is natural, that

$$\alpha(I)(z) = \sum_{j=1}^{n} f(\alpha)(z_j)$$

for $z_j = \mathbf{x}_\Lambda(f_j)(z \parallel S_j)$. Thus the map $\varphi : Z^1(C(x\Lambda)) \longrightarrow \mathbf{x}_\Lambda([1])^*$ just constructed is injective, and it is also linear.

The elements of $\mathbf{x}_\Lambda(\mathbf{1})$ are in bijection with the order ideals of Λ in such a way that a map $\zeta : \Lambda \longrightarrow \wp(\mathbf{1})$ is sent to the order ideal $J(\zeta) = \{\lambda \in \Lambda : \zeta(\lambda) = \varnothing\}$ and an order ideal J is sent to $\zeta(J) : \Lambda \longrightarrow \wp(\mathbf{1})$ such that $\zeta(J)(\lambda) = \{1\}$ if and only if $\lambda \notin J$. For each

$z \in \mathbf{x}_\Lambda(I)$ and $i \in I$, define

$$\chi_I(z,i) = \{\lambda \in \Lambda : i \notin z(\lambda)\}$$

For $\zeta \in P_\Lambda(\mathbf{1})$ let $J = J(\zeta)$ and define for a finite set I and $z \in \mathbf{x}_\Lambda(I)$

$$\alpha_\zeta(I)(z) = \#\{i \in I : \chi_I(z,i) = \chi_{\mathbf{1}}(\zeta,1)\}$$

Then α coincides with the cochain α_J defined earlier, and it is in particular a cocycle. Now, if $\zeta' \in \mathbf{x}_\Lambda(\mathbf{1})$, $\varphi(\alpha_\zeta)(\zeta')$ equals

$$\alpha_\zeta(\{1\})(\zeta') = \#\{i \in \{1\} : \chi_{\mathbf{1}}(\zeta',i) = \chi_{\mathbf{1}}(\zeta,1)\} = \begin{cases} 1 & \text{if } \zeta = \zeta', \\ 0 & \text{else.} \end{cases}$$

This proves that the set $\{\varphi(\alpha_J) : J \in J(\Lambda)\}$ is a basis for the space $\mathbf{x}_\Lambda([1])^*$ and hence that φ is onto, so it is an isomorphism of vector spaces. Since there are no nonzero 1-coboundaries, this concludes the proof of the lemma. ◀

Remark we just defined an isomorphism $H^1(C^*(x\Lambda)) \simeq \hom_k(\mathbf{x}_\Lambda([1]), k)$, and we already know there is an isomorphism $H^0(C^*(x\Lambda)) \simeq \hom_k(\mathbf{x}_\Lambda([0]), k)$. Both results are special cases of Corollary 4.6.

The cohomology of representable species. We show the cohomology algebra of the product species $\mathbf{e}^{\otimes k}$ with their canonical product bicomodule structure can be completely described.

DEFINITION 7.2. For each finite set C, let P^C denote the cosymmetric $\mathbf{e}$-bicomodule species such that

(1) For each finite set I, $\mathbf{x}^C(I)$ is the set of functions $z : I \longrightarrow C$,

(2) For a bijection $\sigma : I \longrightarrow J$ and a function $z : I \longrightarrow C$, $\mathbf{x}^C(\sigma)(f) = f\sigma^{-1}$,

(3) For each finite set, each subset $S \subseteq I$ and each function $z : I \longrightarrow C$, $z \parallel S$ is the restriction of z to S.

We call $\mathbf{x}^C$ the species *represented by the set* C. This gives, in fact, a functor $\mathsf{Set} \longrightarrow \mathsf{Sp}$. Remark that $\mathbf{x}^C$ is canonically isomorphic to $\mathbf{e}^{\otimes k}$ if $\#C = k$, when $\mathbf{e}^{\otimes k}$ is endowed with its canonical product comonoid structure.

DEFINITION 7.3. Let X be a cosymmetric $\mathbf{e}$-bicomodule. Define for each $z \in \mathbf{x}(I)$ and each $\zeta \in \mathbf{x}([1])$ the set

$$\chi_I(z;\zeta) = \{i \in I : \mathbf{x}(f_i)(z \parallel \{i\}) = \zeta\}$$

where f_i is the unique bijection $\{i\} \longrightarrow [1]$. This defines a function $\eta_I(z) : I \longrightarrow \mathbf{x}([1])$ that assigns to each $i \in I$ the element $\zeta \in \mathbf{x}([1])$ if $i \in \chi_I(z;\zeta)$, and defines a function $\chi_I(z) : \mathbf{x}([1]) \longrightarrow \mathbb{N}$ such that $\chi_I(z)(\zeta) = \#\chi_I(z;\zeta)$. We say that $\mathbf{x}$ is 1-*determined* if η_I is an *injection* for every finite set I, and *strongly* 1-*determined* if this is a *bijection* for every finite set I. The proof of Proposition 7.5 will show η is a morphism of symmetric bicomodules from X to $\mathbf{x}^C$ where $C = \mathbf{x}([1])$.

OBSERVATION 7.4. If the species X is 1-*determined* then for every pair of finite sets I, J and every pair of structures $z \in X(I)$, $w \in X(J)$, the following are equivalent.

(1) There is some bijection $\sigma : I \longrightarrow J$ that transports z to w.
(2) The functions $\chi_I(z)$ and $\chi_J(w)$ are equal.

PROPOSITION 7.5. *A linealized cosymmetric* $\mathbf{e}$-*bicomodule is strongly* 1-*determined if and only if it is isomorphic to a representable species.*

PROOF. Two functions $z, w : I \longrightarrow C$ are determined uniquely by their fibres. Our last observation boils down to the fact two functions $z : I \longrightarrow C$ and $w : J \longrightarrow C$ differ by a permutation $\sigma : I \longrightarrow J$ if and only if the fibres $z^{-1}(x)$ and $w^{-1}(x)$ are of the same cardinal for each $x \in C$, and these are precisely the cardinalities of the sets $\chi_I(z;\zeta)$ and $\chi_J(w;\zeta)$ for $\zeta \in X([1])$, for $X([1])$ stands in canonical bijection with the set C. Thus any representable species is strongly 1-determined. Suppose, conversely, that X is strongly 1-determined, and let $X([1]) = C$. We claim that X is isomorphic as a symmetric bicomodule to X^C. By hypothesis the natural transformation $\eta : X \longrightarrow X^C$ is a bijection for each finite set I, so it remains to verify this is a bicomodule morphism.

It suffices to check that for every finite set I, every subset $S \subseteq I$, every structure $z \in X(I)$ and every $\zeta \in X$, $\chi_I(z;\zeta) \cap S = \chi_S(z \parallel S;\zeta)$. But if $i \in S$, $zPsS \parallel \{i\} = z \parallel \{i\}$, so the claim follows. ◄

OBSERVATION 7.6. Note that the species of parts is represented by the set $\{0,1\}$, so by the following theorem $H^*(\wp)$ is canonically isomorphic to the cohomology of the free commutative monoid in two letters. Alternatively, the species of parts is isomorphic to the species associated to the one element poset Λ, and the set of order ideals of Λ has precisely two elements.

THEOREM III.7.7. *Let $\mathbf{x}^C$ be the cosymmetric $\mathbf{e}$-bicomodule represented by a finite set C. There is an isomorphism of complexes*

$$\varphi_C : C^*(\mathbf{x}^C) \longrightarrow C^*(M_C, k)$$

where M_C is the free commutative monoid on the set C and $C^*(M_C, k)$ is the canonical complex that calculates the semigroup cohomology of M_C. It follows that $H^*(\mathbf{x}^C)$ is an exterior algebra with generators in bijection with the elements of C. Moreover, the 1-cocycles that generate $H^*(X^C)$ are given by $\tau_c : X^C \longrightarrow E$ for $c \in C$ where

$$\tau_c(I)(z) = \#z^{-1}(c).$$

PROOF. Suppose $C = \{x_1, \ldots, x_r\}$, and for each non-negative integer n let M_n denote the set of sums $\sum n_i x_i$ with $\sum n_i = n$. There is a bijection $\varphi_n : \pi_0(X^C([n])) \longrightarrow M_n$ that assigns to each function $z : [n] \longrightarrow C$ the element $m = \sum z_i x_i$ where $z_i = \#z^{-1}(x_i)$. This is well defined for if $w = z\sigma$ for a permutation $\sigma \in S_n$, then $w_i = z_i$ for each $i \in [n]$. It is injective, for if two functions $z, w : [n] \longrightarrow X$ are such that $z_i = w_i$ we can (not necessarily in a unique way) define a permutation $\sigma \in S_n$ such that $z = w\sigma$. It is clear this is also surjective. These bijections assemble to give a bijection $\varphi : \pi_0(X^C) \longrightarrow M_C$.

For each $q \in \mathbb{N}_0$, set $M^q = \hom_{Set}(M_C^q, k)$, and define inverse bijections

$$\Phi : C^q(\mathbf{x}^C) \longrightarrow M^q, \quad \Psi : M_C^q \longrightarrow C^q(X^C)$$

as follows. Fix a cochain $\alpha : \mathbf{x}^C \longrightarrow \mathbf{e}^{\otimes q}$ and consider an element $(m_1, \ldots, m_q)$ of $(M_C)^q$. Then $\sum m_i = m \in M_n$ for some n, and by the remarks in the previous paragraph this defines —up to isomorphism— a unique $z : [n] \longrightarrow X$ with $\#z^{-1}(x_j) = n_j$. Write $m_i = \sum n_{ij} x_j$ so that, if we set $\sum_{j=1}^{r} n_{ij} = n_i$, we have $n = n_1 + \cdots + n_q$. Let $(F_1, \ldots, F_q)$ be a decomposition of $[n]$ such that F_i intersects $z^{-1}(x_j)$ in n_{ij} elements. Define, finally

$$\Phi(\alpha)(m_1, \ldots, m_q) = \alpha(F_1, \ldots, F_q)(z).$$

Conversely, consider a function $f : (M_C)^q \longrightarrow k$, and let us define a q-cochain $\Psi(f) : X^C \longrightarrow E^{\otimes q}$. Suppose I is a finite set of size n, $(F_1, \ldots, F_q)$ is a decomposition of I, and pick $z \in X^C(I)$. By the previous paragraph there is a corresponding $m \in M_n$ that depends only on the isomorphism type of z, namely $m = \sum z_j x_j$ where $z_j = z^{-1}(x_j)$.

Let $z_{ij} = \#(S_i \cap z^{-1}(x_j))$ so that $\sum_{i=1}^{q} z_{ij} = z_j$, and set $m_i = \sum z_{ij} x_j$. Define

$$\Psi(f)(F_1,\ldots,F_q)(z) = f(m_1,\ldots,m_q)$$

A tedious verification shows that Ψ, Φ are inverse bijections, and are, in fact, morphisms of cosimplicial objects. The theorem follows. ◀

As we stated earlier, we can now obtain the cohomology of the species $\mathbf{x}_\Lambda$ defined for a poset Λ.

PROPOSITION 7.8. *For any finite poset Λ, the species $\mathbf{x}_\Lambda$ is isomorphic as a cosymmetric bicomodule to the species $X^{J(\Lambda)}$ represented by the set $J(\Lambda)$ of order ideals of Λ.*

PROOF. For any finite set I there are defined inverse bijections between monotone functions $\Lambda \longrightarrow \wp(I)$ and functions $I \longrightarrow J(\Lambda)$ in such a way that $z : \Lambda \longrightarrow \wp(I)$ corresponds to the function $\varphi(z) : I \longrightarrow J(\Lambda)$ such that

$$\varphi(z)(i) = \{\lambda \in \Lambda : i \notin z(\lambda)\}$$

and a function $g : I \longrightarrow J(\Lambda)$ corresponds to the monotone function $\psi(g) : \Lambda \longrightarrow \wp(I)$ such that

$$\psi(g)(\lambda) = \{i \in I : \lambda \notin f(i)\}.$$

This defines, in fact, a natural isomorphism $\eta : \mathbf{x}_\Lambda \longrightarrow \mathbf{x}^{J(\Lambda)}$ which is also a morphism of $\mathbf{e}$-bicomodules, so the claim follows. ◀

This completes the calculation of $H^*(\mathbf{x}_\Lambda)$:

THEOREM III.7.9. *For any finite poset Λ, the cohomology ring $H^*(\mathbf{x}_\Lambda)$ is canonically isomorphic to the exterior algebra on the k-module $H^1(\mathbf{x}_\Lambda)$.*

We also obtain the following corollary.

COROLLARY 7.10. *Suppose X is a cosymmetric $\mathbf{e}$-bicomodule, and suppose further that $X([1])$ a k-module with basis $\{\zeta_1,\ldots,\zeta_r\} = C$. There is a morphism of graded commutative algebras*[3]

$$\Lambda(\zeta_1,\ldots,\zeta_r) \longrightarrow H^*(\mathbf{x})$$

[3]Here $\Lambda(S)$ denotes the exterior algebra with generators S.

induced by the canonical map $\eta : \mathbf{x} \longrightarrow \mathbf{x}^C$ described in Definition 7.3.

Recall that the species represented by a set C on k elements is isomorphic as a cosymmetric bicomodule to the product species $\mathbf{e}^{\otimes k}$, endowed with the canonical product bicomodule structure. The previous calculations give first confirmations that a Künneth-type formula holds for species, that is, the cohomology of the product $\mathbf{x} \otimes \mathbf{y}$ is canonically isomorphic to the graded tensor product of the cohomology of $\mathbf{x}$ and $\mathbf{y}$ in favourable cases.

The cohomology of 1-determined species and Stanley–Reisner rings. If C is a finite set the subbicomodules of the correpresentable species $\mathbf{x}^C$ are 1-determined, and a sub-$\mathbf{e}$-bicomodule of $\mathbf{x}^C$ is completely determined by defining for each finite set I a collection of functions $z : I \longrightarrow C$ that is stable under restrictions.

DEFINITION 7.11. Let C be a finite set. An *abstract simplicial complex* over C is a collection K of finite subsets of C such that

(1) If $x \in C$ then $\{x\} \in K$,
(2) If $\Delta \in K$ and $\Delta' \subseteq \Delta$, then $\Delta' \in K$.

Thus K is an order ideal in the poset of subsets of C that contains every singleton subset of C. We call the elements of K the *simplices* of K and say that Δ is a *q-simplex*, or that it is a simplex of *dimension q*, whenever $\#\Delta = q+1$. We denote by $\dim\Delta$ the dimension of Δ and define the *dimension of K* by $\dim K = \sup\{\dim\Delta : \Delta \in K\}$. If it is necessary we will denote by (C,K) a simplicial complex K defined on C. Remark we include the empty simplex in any simplicial complex.

DEFINITION 7.12. A *simplicial set* is a simplicial object in the category of sets. Because of Lemma 7.3, a simplicial set is defined by specifying a sequence of sets $(X_i)_{i\geqslant 0}$ and respective face and degeneracy maps $\partial_i : X_{q+1} \longrightarrow X_q$, $\sigma_i : X_q \longrightarrow X_{q+1}$ that satisfy the simplicial relations. We will call an element $s \in X_q$ a *q-simplex* of X, and say s is *degenerate* if it is of the form $\sigma_j s'$ for some j and some $s' \in K_{q-1}$. Otherwise, s is said to be a *nondegenerate q-simplex*. We can write $X_q = NX_q \cup DX_q$ where NX_q and DX_q denote the collections of nondegenerate and degenerate q-simplices, respectively. The *dimension* of $\mathbf{x}$ is the largest q such that NX_q is nonempty.

We may associate to every simplicial complex (X,K) a simplicial set $(K_\#, \partial, \sigma)$ as follows. Enlarge to collection of simplices of the complex K by allowing *degenerate q-simplices* which are multisubsets of X of size q where at least one $x \in X$ appears twice.

We will denote by NK_q the collection of *ordered* q-simplices of K which we call the *nondegenerate q-simplices* of $K_\#$, and denote by DK_q the collection of ordered degenerate q-simplices of K. Now let $K_q = NK_q \cup DK_q$, and define $K_\#$ to be the sequence of the sets $K_0, K_1, \ldots$.

The face maps $\partial_j : K_{q+1} \longrightarrow K_q$ are defined so that $\partial_j(xi_0, \ldots, xi_{q+1})$ is the ordered q-simplex obtained by deleting xi_j from $(x_{i_0}, \ldots, x_{i_{q+1}})$, and the degeneracy maps $\sigma_j : Z_q \longrightarrow K_{q+1}$ are defined so that $\sigma_j(x_{i_0}, \ldots, x_{i_{q+1}})$ is the ordered $(q+1)$-simplex obtained by repeating x_{i_j}. One can see inductively that in the above construction DK_{q+1} is the image of $K_q = DK_q \cup NK_q$ under $\sigma_0, \ldots, \sigma_q$.

DEFINITION 7.13. If K is a simplicial complex defined on a finite set C, consider the subspecies $\mathbf{x}_K$ of $\mathbf{x}^C$ such that

(1) For every finite set I, $\mathbf{x}_K(I)$ consists of those functions $z : I \longrightarrow C$ for which $z(I) \in K$,
(2) If $\sigma : I \longrightarrow J$ is a bijection and $z \in \mathbf{x}_K(I)$, $\mathbf{x}_K(\sigma)(z) = z\sigma^{-1}$,
(3) For every finite set I, every subset S of I and $z \in \mathbf{x}_K(I)$, $z \parallel S$ is the restriction of z to S.

Note that if $S \subseteq I$ then $z(S) \subseteq z(I) \in K$, and since K is a simplicial complex, $z(S) \in K$.

DEFINITION 7.14. There is defined a category Simp of finite simplicial complexes that has objects the finite simplicial complexes (C,K) and arrows $(C,K) \longrightarrow (D,L)$ the functions $f : C \longrightarrow D$ such that whenever $\Delta \in K$, it follows that $f(\Delta) \in L$.

The previous construction defines a covariant functor $x_? : \mathsf{Simp} \longrightarrow \mathsf{Sp}$ from the category of abstract simplicial complexes to the category of species in such a way that every simplicial complex (C,K) is assigned the species $\mathbf{x}_K \subseteq \mathbf{x}^C$ and a morphism of simplicial complexes $f : (C,K) \longrightarrow (D,L)$ is assigned the natural transformation $f_* : \mathbf{x}_K \longrightarrow \mathbf{x}_L$ acting by postcomposition on structures.

Like in the previous example, to calculate the cohomology of xL we will introduce an intermediate cosimplicial object obtained by dualizing a simplicial object.

DEFINITION 7.15. If (C,K) is a finite simplicial complex, let Γ_K denote the subset of functions $f : C \longrightarrow \mathbb{N}_0$ of finite support such that the support $\|f\| = \{xi : f(xi) \neq 0\}$ is a simplex of K. In particular, the function that is identically zero has support the empty simplex of K. We define the weight of a function f to be the sum of its values, and denote it by $|f|$.

We now construct the simplicial set that, when properly linearized and dualize, gives the cosimplicial k-module with cohomology equal to $H^*(\mathbf{x}_K)$. Define for each non-negative integer q the set

$$\Gamma_K^{(q)} = \{(f_1,\ldots,f_q) : \sum f_i \in \Gamma_K\}$$

so that, in particular, $\Gamma_K^{(1)} = \Gamma_K$. In general, $\Gamma_K^{(q)}$ is a proper subset of the cartesian product Γ_K^q. The total weight and total support of a tuple $(f_1,\ldots,f_q)$ in $\Gamma_K^{(q)}$ is the weight and support of the sum of its entries. The weight (resp. support) of a tuple is the ordered tuple of the weights (resp. supports) of its entries. Denote by $\Gamma_K^\#$ the sequence of sets just constructed.

THEOREM III.7.16. *There are face and degeneracy maps that endow $\Gamma_K^\#$ with the structure of a cosimplicial set. Moreover, there is an isomorphism of cosimplicial k-modules*

$$\Psi : C^*(\mathbf{x}_K) \longrightarrow \hom_k(k\Gamma_K^\#, k).$$

PROOF. We begin by constructing the face and degeneracy maps. For $i \in \{0,q\}$ we define

$$\partial_0(f_1,\ldots,f_q) = (f_2,\ldots,f_q), \qquad \partial_q(f_1,\ldots,f_q) = (f_1,\ldots,f_{q-1})$$

and for $i \in \{1,\ldots,q-1\}$, we define

$$\partial_i(f_1,\ldots,f_q) = (f_1,\ldots,f_i + f_{i+1},\ldots,f_q).$$

Finally set $\sigma_j(f_1,\ldots,f_q) = (f_1,\ldots,f_{j-1},0,f_j,\ldots,f_q)$ for $j \in \{0,\ldots,q\}$. By a direct calculation, we see these maps satisfy the simplicial relations, and they make $(\Gamma_K^\#,\partial,\sigma)$ into a simplicial set. To obtain the isomorphism Ψ we adapt the proof of Theorem III.7.7. If $\alpha : \mathbf{x}_K \longrightarrow E^{\otimes p}$ is a cochain, we can define a functional $\Psi(\alpha) : \Gamma_K^{(p)} \longrightarrow k$ as follows. To each function $f : C \longrightarrow \mathbb{N}_0$ we can assign a function $z_f : [n] \longrightarrow C$ for some n, that belongs to $\mathbf{x}_K([n])$, and whose isomorphism class depends uniquely on f. Moreover, if $f = (f_1,\ldots,f_p) \in \Gamma_K^{(p)}$, then we can find a function $z_f : [n] \longrightarrow C$ in $\mathbf{x}_K([n])$ and decomposition $(F_1,\ldots,F_p)$ such that the restriction of z_f to F_i corresponds to z_{f_i} under our previous assignment. We then define

$$\Psi(\alpha)(f) = \alpha(F_1,\ldots,F_p)(z_f).$$

Conversely, to each functional $g : \Gamma_K^{(p)} \longrightarrow k$ we can assign a cochain $\Phi(g) : \mathbf{x}_K \longrightarrow E^{\otimes p}$ so that

$$\Phi(g)(F_1,\ldots,F_p)(z) = g(f_1,\ldots,f_p)$$

where $f_i \in \Gamma_K$ corresponds to the restriction of z to F_i, so that in particular the sum of the f_i corresponds to z itself, and lies again in Γ_K. These are well defined inverse bijections, and it is straightforward, albeit very tedious, to check they are cosimplicial morphisms. ◀

As we already did in the previous cases, we will calculate the homology of $kK_\#$ instead of the cohomology of its dual, and then make an appeal to the Universal Coefficient Theorem. To do so, we use, again, a spectral sequence. We can filter $\Gamma_K^{(q)}$ by weight, so that $F_p\Gamma_K^{(q)}$ consists of those tuples of (total) weight at most p. Moreover, we may normalize so that the tuples $(f_1,\ldots,f_q)$ consist of entries with nonzero weight —that is, of nonzero entries. The associated spectral sequence starts at the E^0-page, where $E^0_{p,q}$ is the free k-module with basis the tuples in Γ_K^{p+q} with nonzero entries and total weight p. In particular, we must have $p+q \geqslant 0$, $p \geqslant 0$ and $q \leqslant 0$. Thus our spectral sequence looks again like that of the Figure 1.

Before computing the vertical homology of the complexes $E^0_{p,*}$ we note that in the induced differential $d^0 : E^0_{p,q} \longrightarrow E^0_{p,q-1}$ the first and last differentials strictly decrease the weight of a tuple, so they vanish. Thus our differential is induced from the face maps

$$\partial_i(f_1,\ldots,f_r) = (f_1,\ldots,f_i+f_{i+1},\ldots,f_r).$$

The following observation will allow us to reduce our problem of computing the E^1-page to one independent of the chosen simplicial complex. We will use the notation $K_{p,r}$ for the vector space with basis the tuples $(f_1,\ldots,f_r)$ with weight exactly p, so that the complex $E^0_{p,*}$ reads

$$0 \longrightarrow K_{p,p} \longrightarrow K_{p,p-1} \longrightarrow \cdots \longrightarrow K_{p,1} \longrightarrow 0 \ .$$

OBSERVATION 7.17. Fix $f \in \Gamma_K$. For each r we can consider the subspace $K_{p,r}(f)$ generated by those tuples $(f_1,\ldots,f_r)$ whose sum is exactly f. Because d^0 preserves the sum of a tuple, $K_{p,*}(f)$ is a subcomplex of $K_{p,*}$, and it is immediate that $K_{p,*}$ is the direct sum of $K_{p,*}(f)$ as f ranges through Γ_K. Moreover $K_{p,*}(f)$ is nonzero exactly when f has weight p. Remark that because d^0 preserves the sum of a tuple, a fortiori it preserves the support and weight of a tuple. In particular, if we fix a simplex $\sigma \in K$ and consider $K_{p,r}(\sigma)$ the subspace generated by those tuples with support σ, $K_{p,*}(\sigma)$ is a subcomplex. Moreover, $K_{p,*}(\sigma)$ is zero if σ has dimension at least p: if f has support with size $p+1$, then its weight is at least $p+1$. Finally, the complex $K_{p,*}(\sigma)$ decomposes as the direct sum of $K_{p,*}(f)$ where $\|f\| = \sigma$. Thus only simplices of dimension p or less contribute to $K_{p+1,*}$.

We summarize the above in the following lemma. Note that for $p = 0$ the space $E^0_{0,0}$ is one dimensional with basis the zero function.

LEMMA 7.18. *For every nonzero p there is a finite direct sum decomposition*

$$K_{p,*} = \bigoplus_{\sigma \in K} K_{p,*}(\sigma)$$

where $K_{p,r}(\sigma)$ is generated by those tuples with total support σ. Moreover, the only possibly nonzero summands are of the form $K_{p,*}(\sigma)$ for $\dim \sigma < p$, and for each $\sigma \in K$ there is a direct sum decomposition

$$K_{p,*}(\sigma) = \bigoplus_{\|f\|=\sigma} K_{p,*}(f)$$

where $K_{p,*}(f)$ is generated by those tuples with sum f. ◀

By the above, it suffices to calculate the homology of the complexes $K_{p,*}(f)$. Suppose that $\sigma = \{xi_0, \dots, xi_d\}$ has dimension $d < p$. If f has weight exactly p then f corresponds to a multiset $M(f)$ of size p on σ. Moreover, a tuple $(f_1, \dots, f_r)$ that sums to f is equivalent to a decomposition of the multiset $M(f)$ into the multisets $M(f_i)$, $i \in \{1, \dots, r\}$. One can then check that, under this identification, $K_{p,*}(f) = C_*(\Sigma, M(f))$. As in the case of Σ and L, we deduce that

PROPOSITION 7.19. *The E^1 page of the spectral sequence associated to (C, K) has only one nonzero row, which lies on the p-axis. Moreover $E^1_{p+1,0}$ has a canonical basis indexed by the p-simplices of K, and the differential d^1 is identically zero. Thus the p-th cohomology group of $\mathbf{x}_K$ has dimension equal to the number of $(p-1)$-simplices in K.*

Observe that this generalizes our computation for the correpresentable species $\mathbf{x}^C$ which is $\mathbf{x}_K$ for $K = \wp(X)$. We can also recover the algebraic structure of the cohomology algebra:

THEOREM III.7.20. *The cohomology ring of the* $\mathbf{e}$*-bicomodule* $\mathbf{x}_K$ *is isomorphic to the graded commutative Stanley Reisner ring associated to* K*, which is generated in degree* 1 *by the vertices of* K *and is subject to the relations* $xi^2 = 0$ *for each vertex* $x_i \in K$ *and to the relations that* $x_{i_1} \cdots x_{i_k} = 0$ *whenever* $\{x_{i_1}, \ldots, x_{i_k}\}$ *is not a simplex of* K*. The cocycles that generate* $H^*(\mathbf{x}_K)$ *in degree* 1 *are* $\tau_v : \mathbf{x}_K \longrightarrow E$ *where* $v \in X$ *and*

$$\tau_v(I)(z) = \#z^{-1}(x).$$

PROOF. For a composition $F = (F_1, \ldots, F_k)$ and for $x_1, \ldots, x_k$ not necessarily distinct elements of $\mathbf{x}$, let $\Delta = (x_1, \ldots, x_k)$ and write

$$\tau_\Delta(F)(z) = (\tau_{x_1} \smile \cdots \smile \tau_{x_k})(F)(z) = \prod_{i=1}^{k} \tau_{x_i}(F_i)(z \parallel F_i).$$

If every xi is different and the underlying set to $\Delta = (x_1, \ldots, x_k)$ is not a simplex of K, then this last product is zero: if every $z^{-1}(x_i) \cap F_i$ has at least one element, this implies the image of z contains at least $\{x_1, \ldots, x_k\}$, which cannot be. Because each τ_x has square zero, it follows that the product above is zero for any combination whose underlying set does not produce a simplex in K. Assume now the xi are distinct. We show that if $\{x_1, \ldots, x_k\}$ is a simplex then this cocycle is not a coboundary. To do so, it suffices to check that $\varepsilon(\tau_\Delta)$ is not zero where ε is the antisymmetrization operator defined in Section 4. If $x_1, \ldots, x_k$ is a simplex, for any composition F we can find some function $z : I \longrightarrow \mathbf{x}$ such that $z^{-1}(x_i) = F_k$ – for example, send every element of F_k to x_k. It follows that if ω is not the identity permutation, then $\omega \cdot \tau(F)(z) = 0$, so that for this choice of F and z, we have $\varepsilon\tau(F)(z) = |F_1| \cdots |F_k| \neq 0$. This argument also shows that the various τ_Δ where Δ ranges through d-simplices in K are linearly independent. Indeed, for any other d-simplex Δ' define $z_{\Delta'} : \Delta' \longrightarrow \mathbf{x}$ to be the inclusion, then considering Δ' as a composition of itself in the order the xi appear in $\tau_{\Delta'}$, we have

$$\varepsilon\tau_\Delta(\Delta')(z_{\Delta'}) = \begin{cases} 1 & \text{if } \Delta = \Delta', \\ 0 & \text{if } \Delta \neq \Delta'. \end{cases}$$

Because we already know the dimension of each homogeneous component of $H^*(\mathbf{x}_K)$ is the correct one, this proves that $\{[\![\tau_\Delta]\!] : \Delta \in K\}$ form a basis of $H^*(\mathbf{x}_K)$, that the set $\{[\![\tau_x]\!], x \in X\}$ generates $H^*(\mathbf{x}_K)$ as an algebra and that the relations that hold among these generators are those of the graded Stanley–Reisner ring of K. ◀

Observe that the above implies the simplicial complex K can be completely reconstructed from the cohomology algebra of $\mathbf{x}_K$. Our calculations also show that we may restrict ourselves to the class of *injective* functions whose images are simplices in some simplicial complex (C, K). That is, we have the following corollary.

COROLLARY 7.21. *Let $\mathbf{x}'_K$ denote the subbicomodule of $\mathbf{x}_K$ consisting of injections $z : I \longrightarrow C$ such that $z(I) \in K$. The inclusion $\mathbf{x}'_K \longrightarrow \mathbf{x}_K$ is induces a quasi-isomorphism of complexes $C^*(\mathbf{x}_K) \longrightarrow C^*(\mathbf{x}'_K)$. Remark that $\mathbf{x}'_K$ is such that $\mathbf{x}'_K(J) = 0$ if J has cardinality greater than $\mathbf{x}$, that is, $\mathbf{x}'_K$ is a species of finite length.* ◀

CHAPTER IV

The combinatorial complex

The objective of this chapter is to obtain an alternative and more useful description of the cohomology groups of a species in Sp_k. We show that for every E-bicomodule $\mathbf{x}$ there is a filtration on the complex $C^*(\mathbf{x})$ giving rise to a spectral sequence of algebras which converges to $H^*(\mathbf{x})$. If $\mathbf{x}$ is *weakly projective*, that is, if for each non-negative integer j, $\mathbf{x}(j)$ is a projective kS_j-module, this collapses at the E^1-page. Because we can completely describe this page, this provides us with a complex that calculates $H^*(\mathbf{x})$, and which can be used for effective computations. To be explicit, by this we mean each component of this complex is finitely generated whenever $\mathbf{x}$ has finitely many structures on each finite set, and in that case the differential of an element depends on finite data obtained from it —this is in contrast with the situation of $C^*(\mathbf{x})$. Moreover, the spectral sequence is one of algebras whenever we endow $C^*(\mathbf{x})$ with a cup product arising from a diagonal map Δ, so these remarks apply to the computation of the cup product structure of $H^*(\mathbf{x})$ obtained from Δ, and we exploit this for the cup product we defined in Chapter III.

More generally, similar arguments show there is a functorial spectral sequence of algebras for every connected bimonoid H in Sp_k and every H-bicomodule $\mathbf{x}$, and we discuss this briefly in Chapter V.

We fix some useful definitions we will use in this chapter. Let $\mathbf{x}$ be a species. The *support* of $\mathbf{x}$ is the set of non-negative integers j for which $\mathbf{x}([j])$ is nontrivial. We say $\mathbf{x}$ is *finitely supported* if is has finite support, and that it is *concentrated in cardinal* j if the support of $\mathbf{x}$ is $\{j\}$. The support of a nontrivial species $\mathbf{x}$ is contained in a smallest interval of non-negative integers, whose length we call the *length of* $\mathbf{x}$. The species $\mathbf{x}$ is of *finite type* if $\mathbf{x}([j])$ is a finitely generated k-module for each nonnegative integer j, and it is *finite* if it is both of finite type and finitely supported.

1. The spectral sequence

Let $\mathbf{x}$ be a species in Sp_k and let j be a non-negative integer. We define species $\tau^j\mathbf{x}$ and $\tau_j\mathbf{x}$, which we call the *upper truncation of* $\mathbf{x}$ *at* j and the *lower truncation of* $\mathbf{x}$ *after* j as follows. For every finite set I, we put

$$\tau^j\mathbf{x}(I) = \begin{cases} \mathbf{x}(I) & \text{if } \#I \leqslant j, \\ 0 & \text{else,} \end{cases} \qquad \tau_j\mathbf{x}(I) = \begin{cases} \mathbf{x}(I) & \text{if } \#I \geqslant j, \\ 0 & \text{else.} \end{cases}$$

If $\sigma : I \longrightarrow J$ is a bijection then $(\tau^j\mathbf{x})(\sigma) = \mathbf{x}(\sigma)$ whenever I has at most j elements, while $(\tau^j\mathbf{x})(\sigma)$ is the unique isomorphism $0 \longrightarrow 0$ in the remaining cases. Similarly, $(\tau_j\mathbf{x})(\sigma) = \mathbf{x}(\sigma)$ whenever I has at least j elements, while $(\tau_j\mathbf{x})(\sigma)$ is the unique isomorphism $0 \longrightarrow 0$ in the remaining cases. It is clear both of this constructions depend functorially on $\mathbf{x}$, and that there is a short exact sequence

$$0 \longrightarrow \tau^j\mathbf{x} \longrightarrow \mathbf{x} \longrightarrow \tau_{j+1}\mathbf{x} \longrightarrow 0. \tag{7}$$

By convention, $\tau^j\mathbf{x} = 0$ and $\tau_j\mathbf{x} = \mathbf{x}$ if j is negative. We will write τ_i^j for the composition $\tau_i \circ \tau^j$, which is the same as $\tau^j \circ \tau_i$, and $\mathbf{x}(j)$ instead of τ_j^j; this species is concentrated in cardinal j. This will be of use in Section 2. Observe that we can carry out these constructions in the categories of $\mathbf{h}$-(bi)comodules for any comonoid $\mathbf{h}$ in Sp_k. Precisely, we have the following proposition:

PROPOSITION 1.1. *Let* $\mathbf{h}$ *be a comonoid in* Sp_k*, let* $\mathbf{x}$ *be a left* $\mathbf{h}$*-comodule, and fix a non-negative integer* j*.*

- **T1.** *The truncated species* $\tau^j\mathbf{x}$ *is an* $\mathbf{h}$*-subcomodule of* $\mathbf{x}$ *in such a way that the inclusion* $\tau^j\mathbf{x} \longrightarrow \mathbf{x}$ *is a morphism of* $\mathbf{h}$*-comodules, and*
- **T2.** *the truncated species* $\tau_j\mathbf{x}$ *is uniquely an* $\mathbf{h}$*-comodule in such a way that the morphisms in the short exact sequence* (7) *in* Sp_k *are in fact of* $\mathbf{h}$*-comodules.*

It is clear the above can, first, be extended to $\mathbf{h}$-bicomodules, and second, be dualized to $\mathbf{h}$-modules, and then extended to $\mathbf{h}$-bimodules. This provides a spectral sequence for monoids and modules, which we will not discuss.

PROOF. Denote by λ the coaction of $\mathbf{x}$. To see **T1**, we have to show that $\lambda(\tau^j\mathbf{x}) \subseteq \mathbf{h} \otimes \tau^j\mathbf{x}$, which is immediate, and **T2** is deduced from this: we identify $\tau_j\mathbf{x}$ with the quotient $\mathbf{x}/\tau^j\mathbf{x}$, which inherits an H-comodule structure making the maps in the short exact sequence (7) maps of H-comodules. ◀

In what follows, we will need to identify the comodule structure of $\mathbf{x}(j)$. This is done in the following lemma, whose proof we omit.

LEMMA 1.2. *Let H be a connected comonoid in Sp_k. An H-(bi)comodule concentrated in one cardinal necessarily has the trivial H-coaction.* ◀

Let $\mathbf{x}$ be an E-bicomodule. For each integer p, let $F^pC^*(\mathbf{x})$ be the collection of chains that vanish on $\tau^{p-1}\mathbf{x}$. This is a subcomplex because $\tau^p\mathbf{x}$ is a E-subbicomodule of $\mathbf{x}$, so we have a descending filtration of the complex $C^*(\mathbf{x})$. When there is no danger of confusion, we will write F^pC^* instead of $F^pC^*(\mathbf{x})$. This filtration in $C^*(\mathbf{x})$ induces a filtration on $H^* = H^*(\mathbf{x})$ with $F^pH^*(\mathbf{x}, C) = \operatorname{im}(H^*(F^pC^*) \longrightarrow H^*)$, and we write $E_0(H)$ for the bigraded object with

$$E_0^{p,q}(H) = \frac{F^pH^{p+q}}{F^{p+1}H^{p+q}}.$$

As explained in detail in [McC2001, Chapter 2, §2], this filtration gives rise to a cohomology spectral sequence $(E_r, d_r)_{r\geqslant 0}$. According to the construction carried out there, the E_0-page has

$$E_0^{p,q} = \frac{F^pC^{p+q}}{F^{p+1}C^{p+q}}$$

and differential $d_0^{pq} : E_0^{pq} \longrightarrow E_0^{p,q+1}$ induced by that of $C^*(\mathbf{x})$, and, in particular, we have $E_0^{pq} = 0$ when $p < 0$ or $p + q < 0$. Moreover:

PROPOSITION 1.3. *Let p be an integer.*

(1) There is a natural isomorphism $F^pC^*(\mathbf{x}) \longrightarrow C^*(\tau_p\mathbf{x})$ that induces, in turn, an isomorphism $(E_0^{p*}, d_0^{p*}) \longrightarrow C^{p+*}(\mathbf{x}(p))$, so that

(2) for every integer q, there are isomorphisms $E_1^{pq} \longrightarrow H^{p+q}(\mathbf{x}(p))$, and, viewing this as an identification,

(3) the differential $d_1^{pq} : E_1^{pq} \longrightarrow E_1^{p+1,q}$ is the composition of the connecting homomorphism $H^{p+q}(\mathbf{x}(p)) \longrightarrow H^{p+q+1}(\tau_{p+1}\mathbf{x})$ of the long exact sequence corresponding to the short exact sequence $0 \longrightarrow \mathbf{x}(p) \longrightarrow \tau_p\mathbf{x} \longrightarrow \tau_{p+1}\mathbf{x} \longrightarrow 0$ and the map $H^*(\iota)$ induced by the inclusion $\iota : \mathbf{x}(p+1) \longrightarrow \tau_{p+1}\mathbf{x}$.

PROOF. The exact sequence of E-bicomodules

$$0 \longrightarrow \tau^{p-1}\mathbf{x} \longrightarrow \mathbf{x} \xrightarrow{\pi} \tau_p\mathbf{x} \longrightarrow 0$$

is split in Sp_k, so applying the functor $C^*(?)$ gives an exact sequence

$$0 \longrightarrow C^*(\tau_p\mathbf{x}) \longrightarrow C^*(\mathbf{x}) \longrightarrow C^*(\tau^{p-1}\mathbf{x}) \longrightarrow 0.$$

This gives the desired isomorphism of $F^pC^*(\mathbf{x})$ with $C^*(\tau_p\mathbf{x})$, since the injective map $C^*(\pi,\mathbf{x})$ has image the kernel of $C^*(\iota)$, which is, by definition, $F^pC^*(\mathbf{x})$. This proves the first claim of the proposition.

Similarly, we have a short exact sequence of bicomodules

$$0 \longrightarrow \mathbf{x}(p) \longrightarrow \tau_p\mathbf{x} \longrightarrow \tau_{p+1}\mathbf{x} \longrightarrow 0,$$

also split in Sp_k, and which gives us the exactness of the second row of the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & F^{p+1}C^* \arrow[r] \arrow[d] & F^pC^* \arrow[r] \arrow[d] & E_0^{p*} \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & C^*(\tau_{p+1}\mathbf{x}) \arrow[r] & C^*(\tau_p\mathbf{x}) \arrow[r] & C^*(\mathbf{x}(p)) \arrow[r] & 0
\end{tikzcd}$$

The desired natural isomorphism $E_0^{p,*} \longrightarrow C^*(\mathbf{x}(p))$ is the unique dashed arrow that extends the commutative diagram, and this proves the second claim of the proposition. To prove the last one, we note the diagram above can be viewed as an isomorphism of *exact sequences*, and so the connecting morphisms are also identified. Moreover, the differential at the E_1-page is induced from the connecting morphism of long exact sequence associated to the exact sequence

$$0 \longrightarrow F^{p+1}C^* \longrightarrow F^pC^* \longrightarrow F^pC^*/F^{p+1}C^* \longrightarrow 0$$

and the projection $F^{p+1}C^* \longrightarrow F^{p+1}C^*/F^{p+2}C^*$ which correspond, under our isomorphisms, to the connecting morphism of the short exact sequence

$$0 \longrightarrow \mathbf{x}(p) \longrightarrow \tau_p\mathbf{x} \longrightarrow \tau_{p+1}\mathbf{x} \longrightarrow 0$$

and to the map $C^*(\iota) : C^*(\tau_{p+1}\mathbf{x}) \longrightarrow C^*(\mathbf{x}(p+1))$ induced by the inclusion. ◀

We will prove in the next section that the spectral sequence just constructed converges to $H^*(\mathbf{x})$. A first step towards this is the following result:

PROPOSITION 1.4. *The filtration is bounded above and complete.*

PROOF. Using the identification provided by the isomorphisms $F^pC^* \longrightarrow C^*(\tau_p, \mathbf{x})$ of Proposition 1.3 and the split exact sequences $0 \longrightarrow \tau^p\mathbf{x} \longrightarrow \mathbf{x} \longrightarrow \tau_{p+1}\mathbf{x} \longrightarrow 0$ we are able, in turn, to identify $C^*(\mathbf{x})/F^{p+1}C^*(\mathbf{x})$ with $C^*(\tau^p\mathbf{x})$. In these terms, what the proposition claims is that the canonical map

$$C^*(\mathbf{x}) \longrightarrow \varprojlim C^*(\tau^p\mathbf{x})$$

is an isomorphism, and this is clear: if a cochain vanishes on every $\tau^p\mathbf{x}$ then it is zero, so the map is injective, and if we have cochains $\alpha^p : \tau^p\mathbf{x} \longrightarrow E^{\otimes *}$ that glue correctly, we obtain a globally defined cochain $\alpha : \mathbf{x} \longrightarrow E^{\otimes *}$, so the map is surjective. ◀

PROPOSITION 1.5. *If* $\mathbf{x}$ *vanishes in cardinals above* N*, then the normalized complex* $\bar{C}^*(\mathbf{x})$ *vanishes in degrees above* N*, and, a fortiori, the same is true for* $H^*(\mathbf{x})$.

PROOF. Let $p > N$, consider a p-cochain α in the normalized complex $\bar{C}^*(\mathbf{x})$, and let us show that α vanishes identically. Indeed, if I is a finite set, the map $\alpha(I) : \mathbf{x}(I) \longrightarrow \overline{E}^{\otimes p}(I)$ is zero: if I has more than p elements, its domain is zero because $\mathbf{x}$ vanishes on I, and if I has at most p elements, then its codomain is zero, since there are no compositions of length p of I. ◀

This has two important consequences, the first of which will be thoroughly exploited in the next sections.

COROLLARY 1.6. *Fix an integer* j.

1. *We have* $H^q(\tau^j\mathbf{x}) = 0$ *if* $q > j$.
2. *The* E_1*-page of the spectral sequence lies in a cone in the fourth quadrant.* ◀

Because the E_1-page of the spectral sequence involves the cohomology of the species $\mathbf{x}(p)$ for $p \geqslant 0$, we turn our attention to the cohomology of species concentrated in a cardinal.

2. Computation of the E_1-page

This section is devoted to describing the E_1-page of the spectral sequence, and showing it concentrated in one row —so that the spectral sequence degenerates at the E_2-page— when $\mathbf{x}$ is weakly projective in Sp_k. Recall that by this we mean that, for each non-negative integer j, $\mathbf{x}(j)$ is a projective kS_j-module.

For $j \geqslant 1$ and for each integer $p \geqslant -1$, let $\Sigma_p(j)$ be the collection of compositions of length $p+2$ of $[j]$. We will identify the elements of $\Sigma_{j-2}(j)$ with permutations of $[j]$ in the obvious way. There are face maps $\partial_i : \Sigma_p(j) \longrightarrow \Sigma_{p-1}(j)$ for $i \in \{0,\ldots,p\}$ given by

$$\partial_i(F_0,\ldots,F_i,F_{i+1},\ldots,F_{p+1}) = (F_0,\ldots,F_i \cup F_{i+1},\ldots,F_{p+1})$$

that make the sequence of sets $\Sigma_*(j) = (\Sigma_p(j))_{p\geqslant -1}$ into an augmented semisimplicial set. We write $k\Sigma_*(j)$ for the augmented semisimplicial k-module obtained by linearizing $\Sigma_*(j)$, and $k\Sigma_*(j)'$ for the dual semicosimplicial augmented k-module.

There is an action of S_j on each $\Sigma_p(j)$ by permutation, so that if $\tau \in S_j$ and if $(F_0,\ldots,F_t)$ is a composition of $[j]$, then

$$\tau(F_0,\ldots,F_t) = (\tau(F_0),\ldots,\tau(F_t)).$$

It is straightforward to check the coface maps are equivariant with respect to this action, so $\Sigma_*(j)$ is, in fact, an augmented semisimplicial S_j-set. Consequently, $k\Sigma_*(j)$ and $k\Sigma_*(j)'$ have corresponding S_j-actions compatible with their semi(co)simplicial structures.

This complex $\Sigma_*(j)$ is known in the literature as the Coxeter complex for the braid arrangement, and its cohomology can be completely described.

PROPOSITION 2.1. *The complex associated to $k\Sigma_*(j)'$ computes the reduced cohomology of a $(j-2)$-sphere with coefficients in k, that is, $H^p(k\Sigma(j)') = 0$ if $p \neq j-2$ and $H^{j-2}(k\Sigma_*(j)')$ is the k-module freely generated by the class of the map $\xi_j : k\Sigma_*(j) \longrightarrow k$ such that*

$$\xi_j(\sigma) = \begin{cases} 1 & \text{if } \sigma = \mathrm{id}, \\ 0 & \text{else.} \end{cases}$$

The action of kS_j on $H^{j-2}(k\Sigma_*(j)')$ is the sign representation.

In what follows, $k[j]$ will denote the sign representation of kS_j just described. Note that, when $j = 1$, $S^{j-2} = \varnothing$, and the reduced cohomology of such space is concentrated in degree -1, where it has value k.

PROOF. We sketch a proof, and refer the reader to [**AM2006**] and [**Bro1989**] for details. The braid arrangement $\mathscr{B}_j$ of dimension j in $\mathbb{R}^j$ is the collection of hyperplanes $\{H_{s,t} : 1 \leqslant s < t \leqslant j\}$, with $H_{s,t}$ defined by the equation $x_t = x_s$. This arrangement has rank $j-1$ and its restriction to the hyperplane H with equation $x_1 + \cdots + x_j = 0$ is essential, and defines a triangulation K of the unit sphere $S^{j-2} \subseteq H$. Concretely, the r-dimensional simplices of K are in bijection with compositions of $[j]$ into $r+2$ blocks, so that a composition $F = (F_0, \ldots, F_{r+1})$ corresponds to the r-simplex obtained by intersecting S^{j-2} with the subset defined by the equalities $x_s = x_t$ whenever s,t are in the same block of F and the inequalities $x_s \geqslant x_t$ whenever $t > s$ relative to the order of the blocks of F. It follows that $k\Sigma_*(j)'$ computes the reduced simplicial cohomology of S^{j-2}, and the generator of the top cohomology group is the functional $\xi_j : k\Sigma_*(j) \longrightarrow k$ described in the statement of the proposition. More generally, if $\xi_\sigma : k\Sigma_{j-2}(j) \longrightarrow k$ is the functional that assigns σ to 1 and every other simplex to zero, then $[\![\xi_\sigma]\!] = (-1)^\sigma [\![\xi_j]\!]$. Because the action of S_j on $k\Sigma_{j-2}(j)'$ is such that $\sigma\xi_j = \xi_\sigma$, this proves $H^{j-2}(k\Sigma_*(j)')$ is the sign representation of kS_j. ◀

We can describe the complex that calculates the cohomology of a species concentrated in cardinal j in terms of the Coxeter complex $\Sigma_*(j)$:

PROPOSITION 2.2. *Fix a non-negative integer $j \geqslant 1$, and let $\mathbf{x}$ be an E-bicomodule concentrated in cardinal j. There is an isomorphism of semicosimplicial k-modules*

$$\Psi^* : \bar{C}^*(\mathbf{x}) \longrightarrow \hom_{S_j}(\mathbf{x}(j), k\Sigma_*(j)'[2]).$$

In particular, if $\mathbf{x}(j)$ is a projective kS_j-module, then $H^p(\mathbf{x}) = 0$ when $p \neq j$ and there is an isomorphism

$$\xi : H^j(\mathbf{x}) \longrightarrow \hom_{S_j}(\mathbf{x}(j), k[j]).$$

This isomorphism is such that if $\alpha : \mathbf{x} \longrightarrow E^{\otimes j}$ is a normalized j-cocycle, then

$$\xi([\![\alpha]\!])(z) = \sum_{\sigma \in S_j} (-1)^\sigma \alpha(\sigma)(z)[\![\xi_j]\!], \tag{8}$$

for each $z \in \mathbf{x}(j)$.

If k is a field of characteristic coprime to $j!$ then every kS_j-module is projective by virtue of Maschke's theorem, so the above applies. If k is a field of characteristic zero, then every species $\mathbf{x}$ is weakly projective, and conversely.

PROOF. Since $\mathbf{x}$ is concentrated in cardinal j, a normalized p-cochain $\alpha : \mathbf{x} \longrightarrow \bar{E}^{\otimes p}$ is completely determined by an S_j-equivariant k-linear map $\tilde{\alpha} : \mathbf{x}(j) \longrightarrow \overline{E}^{\otimes p}(j)$. Moreover, $\overline{E}^{\otimes p}(j)$ is a free k-module with basis the tensors $F_1 \otimes \cdots \otimes F_p$ with $(F_1, \ldots, F_p)$ a composition of $[j]$, that is, $\overline{E}^{\otimes p}(j)$ can be identified S_j-equivariantly with $k\Sigma_{p-2}(j)$. Because $\overline{E}^{\otimes p}(j)$ is a free k-module, every k-linear map $\beta : \mathbf{x}(j) \longrightarrow \bar{E}^{\otimes p}(j)$ corresponds uniquely to a map $\beta^t : \mathbf{x}(j) \longrightarrow \bar{E}^{\otimes p}(j)'$ so that $\beta^t(z)(F_1, \ldots, F_p) = \beta(F_1, \ldots, F_p)(z)$. In this way we obtain a map

$$\Psi^* : \bar{C}^*(\mathbf{x}) \longrightarrow \hom_{S_j}(\mathbf{x}(j), k\Sigma_*(j)'[2]),$$

which is clearly an isomorphism of graded k-modules, and this map is compatible with the semicosimplicial structure and S_j-equivariant. The non-trivial observation needed to check this is that the first and last coface maps of $\overline{C}^*(\mathbf{x})$ are zero: this follows from Lemma 1.2, which states $\mathbf{x}$ has trivial coactions, so these maps vanish upon normalization.

Assume now that $\mathbf{x}(j)$ is kS_j-projective, so that the functor $\hom_{S_j}(\mathbf{x}(j), ?)$ is exact. The canonical map

$$\theta : H^*(\hom_{S_j}(\mathbf{x}(j), k\Sigma_*(j)'[2])) \longrightarrow \hom_{S_j}(\mathbf{x}(j), H^*(k\Sigma_*(j)'[2]))$$

is then an isomorphism, and we can conclude by Lemma 2.1 that $H^p(\mathbf{x})$ is zero except for $p = j$, and that we have a canonical isomorphism induced by Ψ and θ

$$\xi : H^j(\mathbf{x}) \longrightarrow \hom_{S_j}(\mathbf{x}(j), k[j]).$$

It remains to prove the last formula. To this end, consider a j-cocycle $\alpha : \mathbf{x}(j) \longrightarrow E^{\otimes j}(j)$. This corresponds under Ψ to the map $\mathbf{x}(p) \longrightarrow k\Sigma_{j-2}(j)'$ that assigns to z the functional $\sum_\sigma \alpha(\sigma)(z)\xi_\sigma$. Passing to cohomology and using the equality $[\![\xi_\sigma]\!] = (-1)^\sigma [\![\xi_j]\!]$ valid in view of Lemma 2.1 for all $\sigma \in S_j$, we obtain the desired formula (8). ◀

COROLLARY 2.3. *If $\mathbf{x}$ is weakly projective, then E_1 is concentrated in the p-axis, where*

$$E_1^{p,0} \simeq \hom_{S_p}(\mathbf{x}(p), k[p]),$$

so that, in particular, the spectral sequence degenerates at E_2.

This motivates us to consider, independently of convergence matters, the complex new $CC^*(\mathbf{x})$ that has $CC^p(\mathbf{x}) = \hom_{S_p}(\mathbf{x}(p), k[p])$ and differentials induced from that of the E_1-page. Although this may not compute $H^*(\mathbf{x})$, it provides us with another invariant for $\mathbf{x}$. We call $CC^*(\mathbf{x})$ the *combinatorial complex of* $\mathbf{x}$. We will give an explicit formula for its differential in Theorem IV.4.5.

PROOF. The above follows for $p \geqslant 1$ by the last proposition, and the case $p = 0$ follows by definition of the E_0-page. ◀

The description of the inverse arrow to ξ will be useful for computations.

LEMMA 2.4. *With the hypotheses of Proposition 2.2, the inverse arrow to ξ is the map*

$$\Theta : \hom_{S_j}(\mathbf{x}(j), k[j]) \longrightarrow H^j(\mathbf{x})$$

that assigns to an S_j-equivariant map $f : \mathbf{x}(j) \longrightarrow k[j]$ the class of any lift F of f according to the diagram

$$\begin{tikzcd} & \mathbf{x}(j) \arrow[dl, dashed, "F"'] \arrow[d, "f"] & \\ k\Sigma_{j-2}(j)' \arrow[r, "\pi"] & k[j] \arrow[r] & 0 \end{tikzcd}$$

In particular, if k is a field of characteristic coprime to $j!$, we can choose F to be the composition of f with the S_j equivariant map $\Lambda : k[j] \longrightarrow k\Sigma_{j-2}(j)'$ such that

$$\Lambda([\![\xi_j]\!]) = \frac{1}{j!} \sum_{\sigma \in S_j} (-1)^\sigma \xi_\sigma.$$

We can now prove, by an easy inductive argument, that the support of the cohomology groups of a weakly projective species of finite length $\mathbf{x}$ is no bigger than the support of $\mathbf{x}$. This is a second step toward proving the convergence of our spectral sequence, which we will completely address in the next section. Concretely:

PROPOSITION 2.5. *Let $\mathbf{x}$ be an E-bicomodule of finite length, which is weakly projective in Sp_k, and let q be a non-negative integer.*

(1) If $\mathbf{x}$ is zero in cardinalities below q, then $H^i(\mathbf{x}) = 0$ for $i < q$.

(2) *In particular, it follows that* $H^p(\tau_q\mathbf{x}) = 0$ *for* $p < q$.

PROOF. Assume $\mathbf{x}$ is a species that vanishes in cardinalities below q, and proceed by induction on the length ℓ of $\mathbf{x}$. The base case in which $\ell = 1$ is part of the content in Proposition 2.2. Indeed, if $\mathbf{x}$ has lenght 1 it is concentrated in some degree p larger than q, and that proposition says $H^j(\mathbf{x}) = 0$ if $j \neq p$.

For the inductive step, suppose $\ell > 1$, and let j be the largest element of the support of $\mathbf{x}$. The long exact sequence corresponding to

$$0 \longrightarrow \tau^{j-1}\mathbf{x} \longrightarrow \mathbf{x} \longrightarrow \tau_j\mathbf{x} \longrightarrow 0.$$

includes the exact segment

$$\underbrace{H^q(\tau_j\mathbf{x})}_{0} \longrightarrow H^q(\mathbf{x}) \longrightarrow \underbrace{H^q(\tau^{j-1}\mathbf{x})}_{0}. \tag{9}$$

The choice of the integer j implies $\tau_j\mathbf{x}$ is of length one, and $\tau^{j-1}\mathbf{x}$ is of length smaller than that of $\mathbf{x}$, so by induction the cohomology groups appearing at the ends of (9) vanish. This proves the first claim, and the second claim is an immediate consequence of it. ◀

PROPOSITION 2.6. *Let* $\mathbf{x}$ *be an E–bicomodule. For every non-negative integer* j*, the projection* $\mathbf{x} \longrightarrow \tau_{j+1}\mathbf{x}$ *induces*

(1) *a surjection* $H^{j+1}(\tau_{j+1}\mathbf{x}) \longrightarrow H^{j+1}(\mathbf{x})$, *and*
(2) *isomorphisms* $H^q(\tau_{j+1}\mathbf{x}) \longrightarrow H^q(\mathbf{x})$ *for* $q > j+1$.

In terms of the filtration on $H^*(\mathbf{x})$, this means that $F^pH^{p+q} = H^{p+q}$ for $q \geqslant 0$.

PROOF. Fix a non-negative integer j and consider the exact sequence

$$0 \longrightarrow \tau^j\mathbf{x} \longrightarrow \mathbf{x} \longrightarrow \tau_{j+1}\mathbf{x} \longrightarrow 0.$$

The associated long exact sequence gives an exact sequence

$$H^{j+1}(\tau_{j+1}\mathbf{x}) \longrightarrow H^{j+1}(\mathbf{x}) \longrightarrow \underbrace{H^{j+1}(\tau^j\mathbf{x})}_{0},$$

and exact sequences

$$\underbrace{H^{q-1}(\tau^j \mathbf{x})}_{0} \xrightarrow{\delta} H^q(\tau_{j+1}\mathbf{x}) \longrightarrow H^q(\mathbf{x}) \longrightarrow \underbrace{H^q(\tau^j \mathbf{x})}_{0},$$

for $q > j+1$, with the zeroes coming from Proposition 1.5. This proves both claims. ◀

3. Convergence of the spectral sequence

The filtration defined on $C^*(\mathbf{x})$ is bounded above, and we have shown it is complete, so it suffices to check the spectral sequence is regular to obtain convergence —see the *Complete Convergence Theorem* in [**Wei1994**, Theorem 5.5.10]. We have proven the spectral sequences degenerates at the E^2-page when $\mathbf{x}$ is weakly projective, and this implies the spectral sequence is regular, so the cited theorem can be applied. We give a mildly more accessible argument to justify convergence, which the reader can compare with the exposition in [**Hat2002**, pp. 137-140] and [**McC2001**, pp. 99-102].

PROPOSITION 3.1. *If* $\mathbf{x}$ *is an* E*-bicomodule that is weakly projective in* Sp_k*, then the group* $H^p(\tau_{q+1}\mathbf{x})$ *vanishes for every integer* $p < q$*.*

In other words, the filtration on $C^*(\mathbf{x})$ is *regular*, that is, for each integer n, we have that $H^n(F^pC^*) = 0$ for large p depending on n; in this case $p > n$ works. This guarantees the spectral sequence is regular, see [**CE1956**, Chapter XV, §4].

PROOF. Let $\mathbf{x}$ be as in the statement. The sequence of inclusions

$$\cdots \longrightarrow \tau^j \mathbf{x} \longrightarrow \tau^{j+1}\mathbf{x} \longrightarrow \cdots \tag{10}$$

gives a tower of cochain complexes $\mathscr{C} = \{C(\tau^j \mathbf{x})\}_{j\geqslant 1}$ of k-modules. We noted, in the proof of Proposition 1.4, that the canonical map $C^*(\mathbf{x}) \longrightarrow \varprojlim_j C^*(\tau^j\mathbf{x})$ is an isomorphism, and furnishes a map

$$\eta : H^*(\mathbf{x}) \longrightarrow \varprojlim_j H^*(\tau^j \mathbf{x}).$$

Let us show that this is an isomorphism. Fix $r \geqslant 0$. The tower of cochain complexes $\mathscr{C}$ satisfies the Mittag-Leffler condition since every arrow in it is onto: every inclusion in (10) is split in Sp_k, so by Theorem A.3.6 there is a short exact sequence

$$0 \longrightarrow \varprojlim_j^1 H^{r-1}(\tau^j \mathbf{x}) \longrightarrow H^r(\mathbf{x}) \xrightarrow{\eta} \varprojlim_j H^r(\tau^j \mathbf{x}) \longrightarrow 0.$$

We need only prove $\varprojlim_j^1 H^{r-1}(\tau^j \mathbf{x}) = 0$, and, to do this, that the tower of abelian groups $\{H^{r-1}(\tau^i \mathbf{x})\}_{i\geqslant 0}$ satisfies the Mittag-Leffler condition: let $\iota(k,j) : H^r(\tau^k \mathbf{x}) \longrightarrow H^r(\tau^j \mathbf{x}))$ be the arrow induced by the inclusion for $k \geqslant j$, and let us show that for each j there is some i such that $\mathrm{image}(\iota(k,j)) = \mathrm{image}(\iota(i,j))$ for every $k \geqslant i$. Fix j, and let us show $i = r+2$ works by considering three cases.

- If $j < r$, then for every $k \geqslant j$ the map $\iota(k,j)$ is zero because its codomain is zero, so the claim is true.
- If $j \geqslant r+1$, then for every $k \geqslant j$, the map $\iota(k,j)$ is an isomorphism. In this case, we have the exact sequence

  $$0 \longrightarrow \tau^j \mathbf{x} \xrightarrow{i} \tau^k \mathbf{x} \xrightarrow{\pi} \tau^k_{j+1} \mathbf{x} \longrightarrow 0$$

  whose corresponding long exact sequence includes the segment

  $$\underbrace{H^r(\tau^k_{j+1}\mathbf{x})}_{0} \longrightarrow H^r(\tau^k \mathbf{x}) \xrightarrow{\iota(k,j)} H^r(\tau^j \mathbf{x}) \longrightarrow \underbrace{H^{r+1}(\tau^k_{j+1}\mathbf{x})}_{0},$$

  with the zeroes explained by Proposition 2.5 and the fact $\tau^k_{j+1}\mathbf{x}$ is zero at cardinals r and $r+1$.
- Finally, suppose $j = r$, and fix $k \geqslant j$. If $k \geqslant r+2$, the map $\iota(k, j+1)$ is an isomorphism, and $\iota(k,j)$ factors as $\iota(j+1,j) \circ \iota(k,j+1)$, so that the image of $\iota(k,j)$ equals the image of $\iota(j+1,j)$.

Fix non-negative integers p and q with $p < q$ as in the statement. For every integer j, the double truncation $\tau^j_{q+1}\mathbf{x}$ is of finite length and begins in degrees greater than q, so that $H^p(\tau^j_{q+1}\mathbf{x}) = 0$ by Proposition 2.5. Because we have just shown that

$$\eta : H^p(\tau_{q+1}\mathbf{x}) \longrightarrow \varprojlim H^p(\tau^j_{q+1}\mathbf{x})$$

is an isomorphism, we can conclude that $H^p(\tau_{q+1}\mathbf{x}) = 0$, as we wanted. ◀

PROPOSITION 3.2. *Suppose* $\mathbf{x}$ *is a weakly projective* E*-bicomodule. There is an isomorphism of bigraded objects* $E_\infty \longrightarrow E_0(H)$*, so that the spectral sequence converges to* H*, and, as it collapses at the* E_1*-page, this gives an isomorphism* $E_2^{p,0} \longrightarrow H^p$.

PROOF. We have already shown that $E_2 = E_\infty$. Moreover, as we observed after Proposition 2.6, we have $F^p H^{p+q} = H^{p+q}$ if $q \geqslant 0$ while, from Proposition 3.1, $H^{p+q}(\tau_p \mathbf{x}) = 0$ when $q < 0$, so that $F^p H^{p+q} = 0$ in this case. This means the only non-trivial filtration quotients are exactly $E_0^{p,0}(H) = H^p$, and that there is an isomorphism

$$E_\infty^{p,0} = E_2^{p,0} \longrightarrow E_0^{p,0}(H)$$

which can be explicitly described as follows. Consider the diagram in Figure 1, built from portions of long exact sequences coming from the split exact sequences

$$0 \longrightarrow \mathbf{x}(i) \longrightarrow \tau_i \mathbf{x} \longrightarrow \tau_{i+1}\mathbf{x} \longrightarrow 0$$

for $i \in \{q-1, q, q+1\}$, and in which the horizontal arrows are the differential d_1 of the E_1-page of our spectral sequence. The maps labelled ι^* in the diagram are injective because the diagonals are exact and there are zeros where indicated, and π^* is surjective by the same reason. We now calculate:

$$\begin{aligned} E_\infty^{p,0} = E_2^{p,0} = \frac{\ker d_1}{\operatorname{im} d_1} = \frac{\ker \delta}{\operatorname{im} \iota^* \delta} &= \frac{\iota^*(H^q(\tau_q \mathbf{x}))}{\iota^* \operatorname{im} \delta} \\ &\simeq \frac{H^q(\tau_q \mathbf{x})}{\operatorname{im} \delta} = \frac{H^q(\tau_q \mathbf{x})}{\ker \pi^*} \\ &\simeq H^q(\tau_{q-1}\mathbf{x}) = E_0^{p,0}(H) \\ &= H^q(\mathbf{x}). \end{aligned}$$

This is what we wanted. ◀

We can summarize the above in the

THEOREM IV.3.3. *If* $\mathbf{x}$ *is an* E*-bicomodule, weakly projective in* Sp_k*, the combinatorial complex* $CC^*(\mathbf{x})$ *computes* $H^*(\mathbf{x})$. ◀

A useful corollary of this is what follows.

$$\begin{tikzcd}
 & & & & H^q(\mathbf{x}(q-1))=0 & & \\
 & 0=H^{q-1}(\tau_q\mathbf{x}) \arrow[dr] & & H^q(\tau_{q-1}\mathbf{x}) \arrow[ur] & & & \\
 & & H^q(\tau_q\mathbf{x}) \arrow[ur, "\pi^*"] \arrow[dr, "\iota^*"] & & & & \\
\cdots \arrow[r] & H^{q-1}(\mathbf{x}(q-1)) \arrow[ur, "\delta"] \arrow[rr, "d_1"] & & H^q(\mathbf{x}(q)) \arrow[rr, "d_1"] \arrow[dr, "\delta"] & & H^{q+1}(\mathbf{x}(q+1)) \arrow[r] & \cdots \\
 & & & & H^{q+1}(\tau_{q+1}\mathbf{x}) \arrow[ur, "\iota^*"] & & \\
 & & & 0=H^q(\tau_{q+1}\mathbf{x}) \arrow[ur] & & &
\end{tikzcd}$$

FIGURE 1. The diagram used in the proof of Proposition 3.2.

COROLLARY 3.4. *If* $\mathbf{x}$ *is an* E*-bicomodule over a field of characteristic zero, then for every integer* q*,*

$$\dim_k H^q(\mathbf{x}) \leqslant \dim_k \hom_{S_q}(\mathbf{x}(q), k[q]),$$

and, in particular, the support of $H^*(\mathbf{x})$ *is contained in that of* $\mathbf{x}$*.* ◂

OBSERVATION 3.5. Fix a nonnegative integer q and a linearized species $\mathbf{x}$. It is useful to note that an element $f \in \hom_{S_q}(\mathbf{x}(q), k[q])$ vanishes on every basis structure $z \in \mathbf{x}(q)$ that is fixed by an odd permutation. This improves the last bound on $\dim_k H^q(\mathbf{x})$ and significantly simplifies computations.

We now have a much better understanding of the derived functors of $F = \hom(?, \mathbf{e})$ in the category of E-bicomodules, and all this was obtained from the canonical injective resolution of the E-bicomodule $\mathbf{e}$. This motivates the following problem:

PROBLEM 3.6. Find a family $\mathbf{x}$ of weakly projective F-acyclic objects, sufficiently large to allow us to resolve arbitrary species.

Solving this problem would allow us, in turn, to calculate the cohomology of species that are not necessarily weakly projective by replacing them with a weakly projective resolution.

4. The differential of the combinatorial complex

The purpose of this section is to give an explicit formula for the differential of the E_1-page of the spectral sequence, equivalently, for the differential of the combinatorial complex, corresponding to a weakly projective **e**-bicomodule **x**. Once this is addressed, we show how to use it to calculate $H^*(\mathbf{x})$ for the species considered in Chapter III. Throughout the section, we fix a weakly projective **e**-bicomodule **x**.

LEMMA 4.1. *The connecting morphism $\delta : H^j(\mathbf{x}(j)) \longrightarrow H^{j+1}(\tau_{j+1}\mathbf{x})$ corresponding to the short exact sequence*

$$0 \longrightarrow \mathbf{x}(j) \longrightarrow \tau_j\mathbf{x} \longrightarrow \tau_{j+1}\mathbf{x} \longrightarrow 0$$

is such that, for a cocycle $\alpha : \mathbf{x}(j) \longrightarrow \bar{\mathbf{e}}^{\otimes j}$, $\delta[\![\alpha]\!] = [\![d\tilde{\alpha}]\!]$ where $\tilde{\alpha} : \tau_j\mathbf{x} \longrightarrow \overline{\mathbf{e}}^{\otimes j}$ is the cochain that extends α by zero away from cardinal j. Therefore, the differential of the E_1-page is such that

$$d_1[\![\alpha]\!] = [\![d\tilde{\alpha}\circ\iota]\!],$$

that is, $d_1[\![\alpha]\!]$ is the class of the restriction of $d\tilde{\alpha}$ to $\mathbf{x}(j+1)$.

PROOF. One follows the construction of the connecting morphism for the diagram of normalized complexes

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
\hom_{\mathsf{Sp_k}}(\tau_{j+1}\mathbf{x},\overline{\mathbf{e}}^{\otimes j}) \arrow[r] \arrow[d] & \hom_{\mathsf{Sp_k}}(\tau_{j}\mathbf{x},\overline{\mathbf{e}}^{\otimes j}) \arrow[r, "\iota^*"] \arrow[d, "d"] & \hom_{\mathsf{Sp_k}}(\mathbf{x}(j),\overline{\mathbf{e}}^{\otimes j}) \arrow[d] \\
\hom_{\mathsf{Sp_k}}(\tau_{j+1}\mathbf{x},\overline{\mathbf{e}}^{\otimes (j+1)}) \arrow[r, "\pi^*"] \arrow[d] & \hom_{\mathsf{Sp_k}}(\tau_{j}\mathbf{x},\overline{\mathbf{e}}^{\otimes (j+1)}) \arrow[r] \arrow[d] & \hom_{\mathsf{Sp_k}}(\mathbf{x}(j),\overline{\mathbf{e}}^{\otimes (j+1)}) \arrow[d] \\
\vdots & \vdots & \vdots
\end{tikzcd}$$

If $\alpha : \mathbf{x}(j) \longrightarrow \mathbf{e}^{\otimes j}$ is a normalized cocycle, and if $\tilde{\alpha} : \tau_j \mathbf{x} \longrightarrow \overline{\mathbf{e}}^{\otimes j}$ extends α by zero then certainly $\iota^* \tilde{\alpha} = \alpha$, and $\tilde{\alpha}$ is normalized, and its restriction to $\mathbf{x}(j)$ is zero. So in fact $d\tilde{\alpha}$ is a cochain

$$d\tilde{\alpha} : \tau_{j+1}\mathbf{x} \longrightarrow \mathbf{e}^{\otimes(j+1)}$$

and it is then its own lift for the map π^*. The lemma follows. ◀

COROLLARY 4.2. *Suppose that $c \in H^j(\mathbf{x}(j))$ is represented by a normalized cocycle $\alpha : \mathbf{x}(j) \longrightarrow \mathbf{e}^{\otimes j}$. Then $d_1(c) \in H^{j+1}(\mathbf{x}(j+1))$ is represented by the normalized cocycle*

$$\gamma : \mathbf{x}(j+1) \longrightarrow \mathbf{e}^{\otimes(j+1)}$$

such that for a permutation σ of a finite set I of $j+1$ elements and $z \in \mathbf{x}(I)$,

$$\begin{aligned}\gamma(\sigma)(z) = \alpha(\sigma(2),\ldots,\sigma(j+1))(z \mathbin{/\!\!/} (I \smallsetminus \sigma(1)))\\ + (-1)^{j+1}\alpha(\sigma(1),\ldots,\sigma(j))(z \mathbin{\backslash\!\!\backslash} (I \smallsetminus \sigma(j+1))).\end{aligned}$$

PROOF. We calculate:

$$\begin{aligned}d\tilde{\alpha}(\sigma(1),\ldots,\sigma(j+1))(z) = \tilde{\alpha}(\sigma(2),\ldots,\sigma(j+1))(z \mathbin{/\!\!/} (I \smallsetminus \sigma(1)))\\ + \sum_{i=1}^{j} (-1)^i \tilde{\alpha}(\sigma(1),\ldots,\sigma(i) \cup \sigma(i+1),\ldots,\sigma(j+1))(z)\\ + (-1)^{j+1}\tilde{\alpha}(\sigma(1),\ldots,\sigma(j))(z \mathbin{\backslash\!\!\backslash} (I \smallsetminus \sigma(j+1))).\end{aligned}$$

Now $\tilde{\alpha}$ equals α on sets of cardinality j so the first and last summands are those of the statement of the corollary, while the sum vanishes, since $\tilde{\alpha}$ vanishes on sets of cardinality different from j. ◀

We have a commutative diagram

$$\begin{tikzcd} H^p(\mathbf{x}(p)) \arrow[r, "\Psi"] \arrow[d, "d_1"] & CC^p(\mathbf{x}) \arrow[d, "\partial"] \\ H^{p+1}(\mathbf{x}(p+1)) \arrow[r, "\Psi"'] & CC^{p+1}(\mathbf{x}) \end{tikzcd}$$

and we have already identified d_1. We now carefully follow the horizontal isomorphisms to obtain the formula for the differential ∂ of the combinatorial complex. The following notation will be useful. If $j \in [p+1]$, let λ_j be the unique order preserving bijection $[p+1] \smallsetminus j \longrightarrow [p]$, and, given a permutation $\sigma \in S_{p+1}$, we write $\sigma \smallsetminus \sigma(j)$ for the permutation $\lambda_{\sigma(j)}\sigma\lambda_j^{-1}$ in S_p. In simple terms, this permutation is obtained by applying $\lambda_{\sigma j}$ to numbers of the list $\sigma 1 \cdots \sigma(\hat{j}) \cdots \sigma(p+1)$. To illustrate, $2143 \smallsetminus 2 = 132$.

LEMMA 4.3. *With the notation above,*

(1) the sign of $\sigma \smallsetminus \sigma(1)$ *is* $(-1)^{\sigma - \sigma(1) - 1}$*, and*

(2) the sign of $\sigma \smallsetminus \sigma(p+1)$ *is* $(-1)^{\sigma + p + 1 - \sigma(p+1)}$.

PROOF. We may obtain the sign of a permutation by counting inversions, that is, if m is the number of inversions in σ, then the sign of σ is $(-1)^m$. By deleting the first number $\sigma(1)$ in σ, we lose $\sigma(1)-1$ inversions coming from those numbers smaller than $\sigma(1)$, and by deleting the last number in σ, we lose $p+1-\sigma(p+1)$ invesions, coming from those numbers larger than $\sigma(p+1)$. ◀

DEFINITION 4.4. Fix a finite set I and a structure $z \in \mathbf{x}(I)$. The *left deck* of z is the set $\operatorname{ldk}(z) = \{z \mathbin{\backslash\!\backslash} (I \smallsetminus i) : i \in I\}$, while the *right deck* of z is the set $\operatorname{rdk}(z) = \{z \mathbin{/\!/} (I \smallsetminus i) : i \in I\}$. If $z \in \mathbf{x}(p)$ and $j \in [p]$, we will write $z'_j \in \mathbf{x}(p-1)$ for $\lambda_j(z \mathbin{\backslash\!\backslash} ([p] \smallsetminus j))$ and $z''_j \in \mathbf{x}(p-1)$ for $\lambda_j(z \mathbin{/\!/} ([p] \smallsetminus j))$.

We now assume k is a field of characteristic zero. With this at hand, we have the following computational result:

THEOREM IV.4.5. *The differential of the combinatorial complex* $CC^*(\mathbf{x})$ *is such that if* $f : \mathbf{x}(p) \longrightarrow k[p]$ *is* S_p*-equivariant, then* $df : \mathbf{x}(p+1) \longrightarrow k[p+1]$ *is the* S_{p+1}*-equivariant map so that for every* $z \in \mathbf{x}(p)$*,*

$$df(z) = \sum_{j=1}^{p+1} (-1)^{j-1} \left(f(z'_j) - f(z''_j) \right).$$

It follows that if $\mathbf{x}$ *is a linearization* $k\mathbf{x}_0$*, the value of* $df(z)$ *for* $f \in CC^p(\mathbf{x})$ *and* $z \in \mathbf{x}_0(p+1)$ *depends only on the left and right decks of* z*. This data is clearly degree-wise finite if* $\mathbf{x}$ *is of finite type.*

PROOF. Fix $f \in CC^p(\mathbf{x})$. Following the correspondence described in Lemma 2.4, the normalized cochain $\alpha : \mathbf{x}(p) \longrightarrow \mathbf{e}^{\otimes p}$ representing f is such that $\alpha(\sigma)(z) = \frac{(-1)^\sigma}{p!} f(z)$

for each $\sigma \in S_p$ and each $z \in \mathbf{x}(p)$. By Lemma 4.1 and its corollary, the differential of α is represented by the cochain $\gamma : \mathbf{x}(p+1) \longrightarrow \mathbf{e}^{\otimes(p+1)}$ such that for $z \in \mathbf{x}(p+1)$ and $\sigma \in S_{p+1}$,

$$\gamma(\sigma)(z) = \alpha(\sigma - \sigma(1))(z'_{\sigma,1}) + (-1)^{p+1}\alpha(\sigma - \sigma(p+1))(z''_{\sigma,p+1}).$$

For brevity, we are writing $z'_{\sigma,i}$ for $z /\!\!/ ([p+1] \smallsetminus \sigma(i))$ and $z''_{\sigma,i}$ for $z \setminus\!\!\setminus ([p+1] \smallsetminus \sigma(i))$. We are also writing $F - F_t$ to denote the composition obtained from F by deleting block F_t. Going back to $CC^{p+1}(\mathbf{x})$ via Proposition 2.2, we obtain that

$$df(z) = \frac{1}{p!} \sum_{\sigma \in S_{p+1}} (-1)^{\sigma} \Big(\alpha(\sigma - \sigma(1))(z'_{\sigma,1}) + (-1)^{p+1}\alpha(\sigma - \sigma(p+1))(z''_{\sigma,p+1}) \Big)$$

and we now split the sum according to the value of $\sigma(1)$ and $\sigma(p+1)$ as follows. If $\sigma(1) = j$, then $z'_{\sigma,1} \in \mathbf{x}([p+1] \smallsetminus j)$, so we may transport this to $[p]$ by means of λ_j: using the notation previous to the statement of the theorem, we have

$$\alpha(\sigma - \sigma(1))(z'_{\sigma,1}) = \alpha(\lambda_j(\sigma - \sigma(1)))(z'_j).$$

Now the sign of the permutation corresponding to the composition $\lambda_j(\sigma - \sigma(1))$, which corresponds to the permutation $\sigma \smallsetminus \sigma(1)$, is $(-1)^{\sigma-(j-1)}$ by Lemma 4.3, so that

$$\alpha(\sigma - \sigma(1))(z'_{\sigma,1}) = (-1)^{\sigma-(j-1)} f(z'_j).$$

Because there are $p!$ permutations σ such that $\sigma(1) = j$ for each $j \in [p+1]$, we deduce that

$$\begin{aligned}
\frac{1}{p!} \sum_{\sigma \in S_{p+1}} (-1)^{\sigma} \alpha(\sigma - \sigma(1))(z'_{\sigma,1}) &= \frac{p!}{p!} \sum_{j=1}^{p+1} (-1)^{\sigma+\sigma-(j-1)} f(z'_j) \\
&= \sum_{j=1}^{p+1} (-1)^{j-1} f(z'_j)
\end{aligned}$$

and this gives the first half of the formula. The second half is completely analogous: the sign $(-1)^{\sigma+p+1}$ partially cancels with $(-1)^{\sigma+p+1-j}$ where $j=\sigma(p+1)$ and we obtain the chain of equalities:

$$\frac{1}{p!}\sum_{\sigma\in S_{p+1}}(-1)^{\sigma}(-1)^{p+1}\alpha(\sigma-\sigma(p+1))(z''_{\sigma,p+1}) = \frac{p!}{p!}\sum_{j=1}^{p+1}(-1)^{j}f(z''_j)$$
$$= -\sum_{j=1}^{p+1}(-1)^{j-1}f(z''_j).$$

This completes the proof of the theorem. ◀

As a consequence of this last theorem, we obtain the following immediate corollaries, which address the structure of the differential of the combinatorial complex for bicomodules that are symmetric or trivial to one side.

COROLLARY 4.6.

D1. *For every symmetric bicomodule* $\mathbf{x}$ *and every nonnegative integer* q *there is an isomorphism* $H^q(\mathbf{x})\simeq \hom_{S_q}(\mathbf{x}(q),k[q])$.

D2. *If* $\mathbf{x}$ *has a trivial left structure, then the differential in* $CC^*(\mathbf{x})$ *is such that for* $f\in CC^p(\mathbf{x})$,

$$d'f(z)=\sum_{j=1}^{p+1}(-1)^{j-1}f(z'_j).$$

There is an analogous statement for for bicomodules with trivial right structure, and we denote the corresponding differential by d''.

5. Some calculations using the combinatorial complex

To illustrate the use of the combinatorial complex, let us show how to recover the calculations of Chapter III and, in doing so, try to convince the reader of the usefulness of the results of this chapter. To begin with, we include a new computation that is greatly simplified with the use of the combinatorial complex.

The species of singletons and suspension. Define the species s of singletons so that for every finite set I, $s(I)$ is trivial whenever I is not a singleton, and is k-free with basis I if I is a singleton. By Lemma 1.2, the species s admits unique right and left $\mathbf{e}$-comodule structures, and thus a unique $\mathbf{e}$-bicomodule structure. By induction, it is easy to check that, for each integer $q \geqslant 1$, the species $s^{\otimes q}$, which we write more simply by s^q, is such that $s^q(I)$ is k-free of dimension $q!$ if I has q elements with basis the linear orders on I, and the action of the symmetric group on I is the regular representation, while $s^q(I)$ is trivial in any other case. By convention, set $s^0 = \mathbf{1}$, the unit species. It follows that the sequence of species $\mathfrak{S} = (s^n)_{n \geqslant 0}$ consists of weakly projective species, and we can completely describe their cohomology groups. They are the analogues of spheres for species, its first property consisting of having cohomology concentrated in the right dimension:

PROPOSITION 5.1. *For each integer $n \geqslant 0$, the species s^n has*

$$H^q(s^n) = \begin{cases} k & \text{if } q = n\,, \\ 0 & \text{else.} \end{cases}$$

PROOF. Fix $n \geqslant 0$. By the remarks preceding the proposition, it follows that $CC^q(s^n)$ always vanishes except when $q = n$, where it equals $\hom_{S_n}(kS_n, k[n])$, and this is one dimensional. Because each s^n is weakly projective, $CC^*(s^n)$ calculates $H^*(s^n)$, and the claim follows. ◀

In Chapter V, we sketch how $s\mathbf{x}$ is a suspension of $\mathbf{x}$, that is, there is an isomorphism of graded k-modules $H^*(s\mathbf{x})[1] \simeq H^*(\mathbf{x})$, at least when $\mathbf{x}$ is weakly projective.

The exponential species. Every structure on a set of cardinal larger than 1 over the exponential species $\mathbf{e}$ is fixed by an odd permutation: if I is a finite set with more than one element, there is a transposition $I \longrightarrow I$, and it fixes $*_I$. It follows that $CC^q(\mathbf{e})$ is zero for $q > 1$, and it is immediate that $CC^0(\mathbf{e})$ and $CC^1(\mathbf{e})$ are one dimensional, while we already know $d = 0$. Thus $HH^q(\mathbf{e})$ is zero for $q > 1$ and is isomorphic to k for $q \in \{0, 1\}$. This confirms the results of Chapter III, Section 1.3.

To illustrate how $s\mathbf{e}$ is truly a suspension of $\mathbf{e}$, let us verify that $H^*(s\mathbf{e})[1] = H^*(\mathbf{e})$. To do so, note that $s\mathbf{e}(I)$ is a free k-module and the elements $i \otimes *_{I \smallsetminus i} \in s(i) \otimes \mathbf{e}(I \smallsetminus i)$ for $i \in I$ form a basis. Moreover, a permutation of S_I fixes the unique basis element in $s(i) \otimes \mathbf{e}(I \smallsetminus i)$ only when it fixes i, so that if I has more than three elements, every element of $s\mathbf{e}(I)$ is fixed by an odd permutation. On the other hand, $s\mathbf{e}(\varnothing) = 0$, $s\mathbf{e}([1]) = k$, and S_2 acts on $s\mathbf{e}([2])$ by permutation, interchanging $s([1]) \otimes \mathbf{e}([2])$ and $s([2]) \otimes \mathbf{e}([1])$. Thus $CC^q(s\mathbf{e})$ vanishes except when $q \in \{1, 2\}$, and in that case it is one dimensional. Finally, observe that because s has a trivial action and $\mathbf{e}$ has a cosymmetric action, $s\mathbf{e}$ has a cosymmetric action. Thus the computation ends here.

The species of linear orders. Recall the species of linear orders L from Chapter I, Section 3, and endow its linearization kL with the $\mathbf{e}$-bicomodule structure described in Chapter II, Section 5. The kS_j-module $kL(j)$ is free of rank one for every $j \geqslant 0$, because S_j acts freely and transitively on the set $L(j)$. It follows that the k-module $\hom_{S_j}(kL(j), k[j])$ is free of rank one, and by virtue of Theorem IV.4.5, the computation ends here: the differential on this combinatorial complex is identically zero. We thus deduce that for every integer $j \geqslant 0$ the k-module $H^j(L)$ is free of rank one, which confirms, again, the results of Chapter III.

The species associated to a simplicial complex. A mildly more elaborate example is the following. Suppose K is a finite simplicial complex over a set C, and $\mathbf{x}_K$ is the associated species of maps $f : I \longrightarrow \mathbf{x}$ with image a simplex of K, as described in Chapter III. There is a left $\mathbf{e}$-comodule structure defined on $\mathbf{x}_K$ such that for $f \in \mathbf{x}_K(I)$ and $S \subseteq I$, $f \setminus\!\!\setminus S$ is the restriction of f to S. This can be turned into an $\mathbf{e}$-bicomodule structure on $\mathbf{x}_K$ in two ways: we may put on $\mathbf{x}_K$ a cosymmetric structure which we write $\mathbf{x}_K^s$, or the trivial right structure, which we denote by $\mathbf{x}_K^t$.

Fix a total order $\preceq$ on C. Consider the complex $(S^*(K), d_K)$ which has $S^j(K)$ the k-module of functions $f : K_j \longrightarrow k$, with K_j the set of j-simplices of K, and differential so that, for $s \in K_{j+1}$,

$$(d_K f)(s) = \sum_{i=0}^{j+1} (-1)^i f(s_i),$$

where s_i is the j-simplex of K obtained by deleting the ith vertex of s. Call $S^*(K)$ the complex of ordered simplicial cochains in K.

PROPOSITION 5.2. *There is an isomorphism of graded k-modules*

$$\Theta : CC^*(\mathbf{x}_K) \longrightarrow S^*(K)[-1]$$

which is an isomorphism of complex in the two cases

$$\Theta^t : CC^*(\mathbf{x}_K^t) \longrightarrow (S^*(K)[-1], d_K) \qquad \Theta^s : CC^*(\mathbf{x}_K^s) \longrightarrow (S^*(K)[-1], 0).$$

As a consequence of this, there are isomorphisms

$$H^*(\mathbf{x}_K^t) \simeq \widetilde{H}^*(SK) \qquad H^*(\mathbf{x}_K^s) \simeq S^*(K)[-1]$$

where SK is the suspension of K.

PROOF. For $j \geqslant 0$ and $s \in K_{j-1}$, let $z_s : [j] \longrightarrow C$ be the unique monotone function with image s, and define $\Theta^j : CC^j(\mathbf{x}_K) \longrightarrow S^{j-1}(K)$ as follows: if $\varphi : \mathbf{x}_K(j) \longrightarrow k[j]$ is equivariant, set $\Theta^j(\varphi)(s) = \varphi(z_s)$ for $s \in K_{j-1}$. It is easy to see this is an isomorphism using Observation 3.5, which in this case tells us an element in $CC^j(\mathbf{x}_K)$ vanishes on non-injective functions. A direct calculation using the formulas for the differential of CC^* in each case, given in Theorem IV.4.5, and the definition of d_K show this is an isomorphism of complexes. The last claim of the proposition follows immediately from this. ◀

The species of partitions. The species of partitions Π assigns to each finite set I the collection $\Pi(I)$ of partitions $\mathbf{x}$ of I, that is, families $\{X_1, \dots, X_t\}$ of disjoint non-empty subsets of I whose union is I. There is a left $\mathbf{e}$-comodule structure on Π defined as follows: if $\mathbf{x}$ is a partition of I and $S \subset I$, $\mathbf{x} \mathbin{\backslash\!\backslash} S$ is the partition of S obtained from the non-empty blocks of $\{x \cap S : x \in \mathbf{x}\}$. There is an inclusion $\mathbf{e} \longrightarrow \Pi$ as described in Chapter II §5.

PROPOSITION 5.3. *The cohomology group $H^0(\Pi)$ is free of rank one, and $H^1(\Pi)$ is free of rank one generated by the cardinality cocycle. In fact, the inclusion $\mathbf{e} \longrightarrow \Pi$ induces an isomorphism $CC^*(\Pi) \longrightarrow CC^*(\mathbf{e})$.*

PROOF. A partition of a set with at least two elements is fixed by a transposition, and this implies, in view of Observation 3.5, that $CC^j(\Pi) = 0$ for $j \geqslant 2$. On the other hand, $CC^0(\Pi)$ and $CC^1(\Pi)$ are both k-free of rank one, and we already know from Proposition IV.4.5 that the differential of $CC^*(\Pi)$ is zero. This proves both claims. ◀

The species of compositions. The species of compositions Σ is the non-abelian analogue of the species of partitions Π. Let us recall its construction: the species of compositions Σ assigns to each finite set I the set $\Sigma(I)$ of compositions of I, that is, ordered tuples $(F_1, \dots, F_t)$ of disjoint non-empty subsets of I whose union is I. This has a standard left $\mathbf{e}$-comodule structure such that if $F = (F_1, \dots, F_t)$ is a composition of I and $S \subseteq I$, $F \mathbin{\backslash\!\backslash} S$ is the composition of S obtained from the tuple $(F_1 \cap S, \dots, F_t \cap S)$ by deleting empty blocks. We view Σ as an $\mathbf{e}$-bicomodule with its cosymmetric structure.

PROPOSITION 5.4. *The morphism $L \longrightarrow \Sigma$ induces an isomorphism $H^*(\Sigma) \longrightarrow H^*(L)$ and, in fact, an isomorphism $CC^*(\Sigma) \longrightarrow CC^*(L)$.*

PROOF. It suffices that we prove the second claim, and, since $C^*(?)$ is a functor, that for a fixed integer q, the map $CC^q(\Sigma) \longrightarrow CC^q(L)$ is an isomorphism of modules. This follows from Observation 3.5: a decomposition F of a set I is fixed by a transposition as soon as it has a block with at least two elements, and therefore an element of $CC^q(\Sigma)$ vanishes on every composition of $[q]$, except possibly on those into singletons. Thus the surjective map $CC^*(\Sigma) \longrightarrow CC^*(L)$ is injective. ◀

The species of graphs. We have already defined the species Gr of graphs along with its cosymmetric $\mathbf{e}$-bicomodule structure. We have the following result concerning the cohomology groups of Gr:

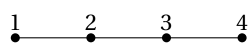
THEOREM IV.5.5. *If k is of characteristic zero then, for each non-negative integer $p \geqslant 0$, $\dim_k H^p(\mathsf{Gr})$ equals the number of isomorphism classes of graphs on p vertices with no odd automorphisms, namely,*

$$1, \quad 1, \quad 0, \quad 0, \quad 1, \quad 6, \quad 28, \quad 252, \quad 4726, \quad 150324, \quad \ldots$$

This sequence is [**Slo2017**, A281003].

PROOF. Since the structure on Gr is cosymmetric, the differential of $CC^*(\mathsf{Gr})$ vanishes, and Observation 3.5 tells us $CC^q(\mathsf{Gr})$ has dimension as in the statement of the proposition. The tabulation of the isomorphism classes of graphs in low cardinalities can be done with the aid of a computer —we refer to Brendan McKay's calculation [**McK2017**] for the final result— and then filter out those graphs with odd automorphisms. ◀

We can exhibit cocycles whose cohomology classes generate $H^1(\mathsf{Gr})$ and $H^4(\mathsf{Gr})$: in degree one, we have the cardinality cocyle κ, and in degree four, the normalized cochain $p^4 : \mathsf{Gr} \longrightarrow \mathbf{e}^{\otimes 4}$ such that for a decomposition $F \vdash_4 I$, and a graph g with vertices on I, $p^4(F_1, F_2, F_3, F_4)(g)$ is the number of inclusions $\zeta : p_4 \longrightarrow g$, where p_4 is the graph



and $\zeta(i) \in F_i$ for $i \in [4]$. One can check this cochain is in fact a cocycle, and it is normalized by construction.

6. Multiplicative structure of the spectral sequence

We have defined a complete descending filtration

$$C^*(\mathbf{x}) \supseteq F^0C^*(\mathbf{x}) \supseteq \cdots \supseteq F^pC^*(\mathbf{x}) \supseteq F^{p+1}C^*(\mathbf{x}) \supseteq \cdots$$

on $C^*(\mathbf{x})$ where $F^pC^*(\mathbf{x})$ consists of those cochains that vanish on $\tau^p\mathbf{x}$. Assume now that $\mathbf{x}$ is a linearized comonoid of the form $k\mathbf{x}_0$, so that there is a cup product defined on $C^*(\mathbf{x})$, as detailed in Chapter III, Section 3. Remark that the proof of the following proposition adapts immediately to any cup product on $C^*(\mathbf{x})$ induced from a diagonal map $\mathbf{x} \longrightarrow \mathbf{x}\square\mathbf{x}$.

PROPOSITION 6.1. *The cup product on $C^*(\mathbf{x})$ is compatible with the filtration, in the sense that, for every two non-negative integers p and p', we have that $F^p \smile F^{p'} \subseteq F^{p+p'}$.*

PROOF. Consider cochains $\alpha \in F^p$ and $\beta \in F^{p'}$. Then $\alpha \smile \beta \in F^{p+p'}$ by a pigeonhole argument: if $F = (F', F'')$ is a decomposition of a finite set with $p + p'$ elements, then F' is a decomposition of a set with at most p elements or F'' is a decomposition of a set with at most p' elements, and the formula

$$(\alpha \smile \beta)(F)(z) = \alpha(F')(z \mathbin{\backslash\!\backslash} F')\beta(F'')(z \mathbin{/\!/} F'')$$

then makes it evident that $\alpha \smile \beta$ is an element of $F^{p+p'}$. ◀

It follows from this proposition that the cup product descends to a product

$$\frac{F^pC}{F^{p+1}C} \otimes \frac{F^{p'}C}{F^{p'+1}C} \longrightarrow \frac{F^{p+p'}C}{F^{p+p'+1}C}$$

so we obtain a multiplicative structure $? \smile ? : E_0^{pq} \times E_0^{p'q'} \longrightarrow E_0^{p+p',q+q'}$ induced on the E_0-page of the spectral sequence. This induces in turn a multiplicative structure on our spectral sequence $(E_r, d_r)_{r\geqslant 0}$. Because this spectral sequence degenerates at E^2, we can compute the cup product in $H^*(\mathbf{x})$ from the combinatorial complex $CC^*(\mathbf{x})$. We describe how to do so in explicit terms.

If S is a subset of $[n] = \{1, \ldots, n\}$ with $m \leqslant n$ elements, and if σ is a permutation of S, we regard σ as a permutation of $[m]$ by means of the unique order preserving bijection $\lambda_S : S \longrightarrow [m]$. We say (σ^1, σ^2) is a (p, q)-*shuffle* of a finite set I with $p + q$ elements whenever σ^1 is a permutation of a p-subset S of I, σ^2 is a permutation of a

q-subset T of I, and $S \cap T = \varnothing$. Call (S,T) the *associated composition* of such a shuffle. If (S,T) is a composition of $[n]$, we will write $\operatorname{sch}(S,T)$ for the *Schubert statistic* of (S,T), which counts the number of pairs $(s,t) \in S \times T$ such that $s < t$ according to the canonical ordering of $[n]$. Our result is the following

THEOREM IV.6.2. *The cup product* $? \smile ? : CC^p(\mathbf{x}) \otimes C^q(\mathbf{x}) \longrightarrow CC^{p+q}(\mathbf{x})$ *is such that for equivariant maps* $f : \mathbf{x}(p) \longrightarrow k[p]$ *and* $g : \mathbf{x}(q) \longrightarrow k[q]$, *and* $z \in \mathbf{x}(p+q)$,

$$(f \smile g)(z) = \sum_{(S,T) \vdash [p+q]} (-1)^{\operatorname{sch}(S,T)} f(\lambda_S(z \mathbin{\backslash\!\backslash} S)) g(\lambda_T(z \mathbin{/\!/} T))$$

where the sum runs through decompositions of $[p+q]$ *with* $\#S = p$ *and* $\#T = q$.

Before giving the proof, we begin with a few preliminary considerations. First, consider a (p,q)-shuffle (σ^1,σ^2) of $[p+q]$, with associated composition (S,T), and let σ be the permutation of $[p+q]$ obtained by concatenating σ^1 and σ^2.

LEMMA 6.3. *For any* $\sigma \in S_{p+q}$ *and any* (p,q)*-composition* (S,T) *of* $[p+q]$,

(1) the sign of σ *is* $(-1)^{\sigma^1+\sigma^2+\operatorname{sch}(S,T)}$, *and*
(2) $(-1)^{\operatorname{sch}(S,T)} = (-1)^{\operatorname{sch}(T,S)+pq}$.

PROOF. Indeed, by counting inversions, it follows that the number of inversions in σ is precisely $\operatorname{inv}\sigma^1 + \operatorname{inv}\sigma^2 + \operatorname{sch}(S,T)$, which proves the first assertion. The second claims follows from the first and the fact $\sigma^1\sigma^2$ and $\sigma^2\sigma^1$ differ by exactly pq transpositions. ◀

Recall that if $\alpha : \mathbf{x}(p) \longrightarrow \mathbf{e}^{\otimes p}$ is a cochain, we associate to it the equivariant map $f : \mathbf{x}(p) \longrightarrow k[p]$ such that $f(z) = \alpha(\nu_p)(z)$ where $\nu_p = \sum_{\sigma \in S_p} (-1)^\sigma \sigma$ is the antisymmetrization element. Conversely, given such an equivariant map, we associate to it the cochain $\alpha : \mathbf{x}(p) \longrightarrow \mathbf{e}^{\otimes p}$ such that $\alpha(\sigma)(z) = \frac{(-1)^\sigma}{p!} f(z)$. We now proceed to the proof of Theorem IV.6.2.

PROOF. To calculate a representative of the class of $f \smile g$, we lift first lift the maps $f : \mathbf{x}(p) \longrightarrow k[p], g : \mathbf{x}(q) \longrightarrow k[q]$ to cochains $\alpha : \mathbf{x} \longrightarrow \mathbf{e}^{\otimes p}, \beta : \mathbf{x} \longrightarrow \mathbf{e}^{\otimes q}$ that are supported in $\mathbf{x}(p)$ and $\mathbf{x}(q)$ respectively, and represent f and g according to the correspondence in the previous paragraph. As in formula (4), we compute for any decomposition (F^1,F^2) of a finite set I and any $z \in \mathbf{x}(I)$ that

$$(\alpha \smile \beta)(F^1,F^2)(z) = \alpha(F^1)(z \mathbin{\backslash\!\backslash} F^1) \beta(F^2)(z \mathbin{/\!/} F^2).$$

Now consider $z \in \mathbf{x}(p+q)$. If σ is a permutation of $[p+q]$, write (σ^1,σ^2) for the (p,q)-shuffle obtained by reading $\sigma(1)\cdots\sigma(p)$ as a permutation of $S_\sigma = \{\sigma(1),\ldots,\sigma(p)\}$ and by reading $\sigma(p+1)\cdots\sigma(p+q)$ as a permutation of $T_\sigma = \{\sigma(p+1),\ldots,\sigma(p+q)\}$. Then

$$\begin{aligned}(f\smile g)(z) &= \sum_{\sigma\in S_{p+q}} (-1)^\sigma (\alpha\smile\beta)(\sigma)(z)\\ &= \sum_{\sigma\in S_{p+q}} (-1)^\sigma \alpha(\sigma^1)(z\mathbin{\backslash\!\backslash} S_\sigma)\beta(\sigma^2)(z\mathbin{/\!/} T_\sigma)\end{aligned}$$

Fix a composition (S,T) of $[p+q]$. In the sum above, the permutations σ with $(S_\sigma,T_\sigma) = (S,T)$ are the (p,q)-shuffles with associated composition (S,T). We may then replace the sum throughout S_{p+q} with the sum throughout (p,q)-compositions (S,T) of $[p+q]$ and in turn with the sum throughout shuffles (σ^1,σ^2) of (S,T). This reads

$$(f\smile g)(z) = \sum_{(S,T)\vdash[p+q]}\sum_{(\sigma^1,\sigma^2)} (-1)^{\sigma^1\sigma^2}\alpha(\sigma^1)(z\mathbin{\backslash\!\backslash} S)\beta(\sigma^2)(z\mathbin{/\!/} T).$$

We now note that $\alpha(\sigma^1)(z\mathbin{\backslash\!\backslash} S) = \alpha(\lambda_S(\sigma^1))(\lambda_S(z\mathbin{\backslash\!\backslash} S))$, that the sign of $\lambda_S(\sigma^1)\in S_p$ is $(-1)^{\sigma^1}$, and that the same considerations apply to β, so we obtain that

$$(f\smile g)(z) = \frac{1}{p!q!}\sum_{(S,T)\vdash[p+q]}\sum_{(\sigma^1,\sigma^2)} (-1)^{\sigma^1\sigma^2+\sigma^1+\sigma^2} f(\lambda_S(z\mathbin{\backslash\!\backslash} S))g(\lambda_T(z\mathbin{/\!/} T)).$$

Using Lemma 6.3 finishes the proof: the sum $\sum_{(\sigma^1,\sigma^2)}(-1)^{\sigma^1\sigma^2+\sigma^1+\sigma^2}$ consists of $p!q!$ instances of $(-1)^{\mathrm{sch}(S,T)}$. ◀

Suppose now that $\mathbf{x}$ is a symmetric $\mathbf{e}$-bicomodule. Then Theorem IV.4.5 proves the differential in $CC^*(\mathbf{x})$ is trivial, while Lemma 6.3 along with Proposition IV.6.2 prove that the cup product in $CC^*(\mathbf{x})$ is graded commutative. We obtain the

THEOREM IV.6.4. *Suppose that* $\mathbf{x}$ *is a cosymmetric* $\mathbf{e}$*-bicomodule. Then* $CC^*(\mathbf{x})$ *is isomorphic to the cohomology algebra* $H = H^*(\mathbf{x})$ *via the isomorphism of algebras* $E_2 \longrightarrow E_0(H)$. *In particular,* $H^*(\mathbf{x})$ *is graded commutative.* ◀

To illustrate, take $\mathbf{x}$ to be the species of linear orders. Each $CC^j(\mathbf{x})$ is one dimensional generated by the map $f_j : L(j)\longrightarrow k$ that assigns $\sigma\longmapsto(-1)^\sigma$. A calculation, which we omit, shows

PROPOSITION 6.5. *The algebra $CC^*(L)$ is generated by the elements f_1 and f_2, so that if f_p is the generator of $CC^p(L)$, we have*

$$f_{2p} \smile f_{2q} = \binom{p+q}{p} f_{2(p+q)}, \qquad f_1 \smile f_{2p} = f_{2p+1}, \qquad f_1 \smile f_{2p+1} = 0.$$

These relations exhibit $H^*(L)$ as a tensor product of a divided power algebra and an exterior algebra. ◀

For a second example, consider Gr with its cosymmetric **e**-bicomodule structure. We already know H^4 is one dimensional, and the functional $p^4 : \mathsf{Gr}(4) \longrightarrow k[4]$ that assigns the 4-path to 1 and every other graph on four vertices to zero is a generator of CC^4. Even more can be said: our formula for the cup product and induction shows that for each $n \geqslant 1$, the product f^n is nonzero on the graph that is the disjoint union of n paths p_4, so that H^{4n} is always nonvanishing for $n \geqslant 1$. Hence the cohomology algebra $H^*(\mathsf{Gr})$ contains both an exterior algebra in degree 1 and a polynomial algebra in degree 4.

CHAPTER V

Future work and problems

In this brief chapter we collect some problems and future projects related to the work in this thesis.

Classifying space of a species. In Chapter IV we showed that we can associate to every $\mathbf{e}$-bicomodule $\mathbf{x}$ a complex $CC^*(\mathbf{x})$. It is important to observe the differentials d' and d'' described in that chapter make $CC^*(\mathbf{x})$ into a cubical k-module: a cubical k-module Q is a non-negatively graded k-module endowed with maps $\lambda_i^\varepsilon : Q_n \longrightarrow Q_{n-1}$ for each $i \in \{1,\dots,n\}$ and $\varepsilon \in \{0,1\}$ that satisfy the *cubical relations*:

$$\text{if } 1 \leqslant i < j \leqslant n \text{ and } \varepsilon, \varepsilon' \in \{0,1\}, \text{ then } \lambda_i^\varepsilon \circ \lambda_j^{\varepsilon'} = \lambda_{j-1}^{\varepsilon'} \circ \lambda_i^\varepsilon.$$

If $(Q, \lambda^0, \lambda^1)$ is a cubical k-module, its *associated chain complex*, which is also denoted by Q, has differential $d : Q_n \longrightarrow Q_{n-1}$ defined by

$$d = \sum_{i=1}^{n} (-1)^i (\lambda_i^0 - \lambda_i^1).$$

If $\mathbf{x}$ is a linearization $k\mathbf{x}_0$, the cubical k-module $CC^*(\mathbf{x})$ is obtained by applying certain hom-functors to a cubical set with components $\{\mathbf{x}_0(q)\}_{q\geqslant 0}$ and restriction maps λ^0, λ^1 obtained from the left and right coactions on $\mathbf{x}$. This motivates us to construct, from this cubical set, a topological space $\mathsf{B}\mathbf{x}$ obtained as a "geometrical realization" of this complex, in such a way that the cohomology of $\mathsf{B}\mathbf{x}$ with coefficients in k is exactly $H^*(\mathbf{x})$, so that $\mathsf{B}\mathbf{x}$ serves as a "classifying space" for $\mathbf{x}$. Having this would make available, in our context, the plethora of operations that can be performed on spaces, such as suspending it, considering its loopspace and obtaining fibrations with base or total space $\mathsf{B}\mathbf{x}$, for example.

The suspension functor and cohomology. In Chapter IV we defined a species s and proved the products $s^{\otimes j}$ are "spheres" for the cohomology functor H^*. Moreover, we checked that $s \otimes \mathbf{e}$ has the same cohomology as $\mathbf{e}$, but shifted one degree up. This motivates us to check whether $s \otimes ?$ acts as a suspension for $H^*(?)$. Assume that $\mathbf{x}$ is weakly projective, so we may use $CC^*(\mathbf{x})$ to compute $H^*(\mathbf{x})$. We claim that $CC^*(s\mathbf{x})$ identifies with $CC^*(\mathbf{x})[-1]$. Indeed, for this it suffices to note, first, that $(s\mathbf{x})(n)$ is isomorphic, as an kS_n-module, to the induced representation $k\otimes\mathbf{x}(n-1)$ from the inclusion $S_1\times S_{n-1}\hookrightarrow S_n$, and second, that the restriction of the sign representation of S_n under this inclusion is the sign representation of S_{n-1}, so that:

$$\begin{aligned}
\hom_{S_n}((s\mathbf{x})(n),k[n]) &= \hom_{S_n}(\mathrm{Ind}_{S_1\times S_{n-1}}^{S_n}(k\otimes\mathbf{x}(n-1)),k[n]))\\
&\simeq \hom_{S_1\times S_{n-1}}(k\otimes\mathbf{x}(n-1),\mathrm{Res}_{S_1\times S_{n-1}}^{S_n}\, k[n])\\
&\simeq \hom_{S_{n-1}}(\mathbf{x}(n-1),k[n-1]).
\end{aligned}$$

A bit more of a calculation shows the differentials are the correct ones. By induction, of course, we obtain that $s^j\mathbf{x}$ has the cohomology of $\mathbf{x}$, only moved j places up.

The cup product structure for a diagonal map. Let $\mathbf{x}$ be an $\mathbf{e}$-bicomodule. We defined a cup product in Chapter III on the complex $C^*(\mathbf{x})$ when the bicomodule structure on $\mathbf{x}$ is defined on a linearized species. One can show that this cup product can be constructed instead in terms of a diagonal map $\Delta:\mathbf{x}\longrightarrow\mathbf{x}\square\mathbf{x}$ afforded by a comonoid structure on $\mathbf{x}$. The point of proceeding in this way is that we may define cup products on the cohomology of $\mathbf{e}$-bicomodules that admit an $\mathbf{e}$-bicolinear map $\Delta:\mathbf{x}\longrightarrow\mathbf{x}\square\mathbf{x}$. It would be desirable to determine whether different nontrivial diagonal maps determine different cup products on $H^*(\mathbf{x})$: it may very well happen that two diagonals Δ and Δ', although inducing different cup products in $C^*(\mathbf{x})$, descend to the same product in $H^*(\mathbf{x})$. If this is not the case, one may obtain different algebra structures on $H^*(\mathbf{x})$, that may, for example, aid in computations.

The spectral sequence of a connected bimonoid. One can extend the work done in the first two sections of Chapter IV by replacing $\mathbf{e}$ with any linearized connected bimonoid $\mathbf{h}$ in Sp_k along the following lines. Let $\mathbf{x}$ be an $\mathbf{h}$-bicomodule. The filtration $F^pC^*(\mathbf{x},\mathbf{h})$ of $C^*(\mathbf{x},\mathbf{h})$ by the subcomplexes

$$\{\alpha:\mathbf{x}\longrightarrow\mathbf{h}^{\otimes *}:\alpha \text{ vanishes on } \tau^{p-1}\mathbf{x}\}$$

is natural with respect to $\mathbf{h}$, and it is complete and bounded above. This yields a spectral sequence for each connected comonoid H starting at $E_0^{p,*}=C^{p+*}(\mathbf{x}(p),\mathbf{h})$ with first page concentrated in a cone in the fourth quadrant, and to prove this is convergent in the sense of [**Wei1994**], it suffices we show this spectral sequence is regular. It should be possible to prove the filtration giving rise to such spectral sequence is regular, which is equivalent to the statement that $H^p(\tau_j \mathbf{x}, \mathbf{h}) = 0$ for large values of j. This is trivially true if $\mathbf{x}$ is of finite length because in such case $\tau_j \mathbf{x} = 0$ for large values of j. Independent of convergence matters, we can identify its first page. Indeed, for each natural number q, write $\langle \mathbf{h}; q\rangle$[1] for the cosimplicial k-module

$$0 \longrightarrow \mathbf{h}^{\otimes 0}([q]) \longrightarrow \mathbf{h}^{\otimes 1}([q]) \longrightarrow \cdots \longrightarrow \mathbf{h}^{\otimes j}([q]) \longrightarrow \cdots$$

with coface maps and codegeneracies induced by Δ and ε, and write $\langle \bar{\mathbf{h}}; q\rangle$ for the corresponding normalized complex of $\langle \mathbf{h}; q\rangle$. Often we can find a topological space $\langle \mathscr{H}; q\rangle$ whose cohomology coincides with that of $\langle \mathbf{h}; q\rangle$. Denote by $\mathscr{H}^{p,q}$ the cohomology groups $H^{p+q}(\langle \mathbf{h}; p\rangle)$, which are all S_p-modules. If $\mathbf{x}$ is weakly projective, the arguments outlined in Section 2 of Chapter IV show that the E_1-page of the spectral sequence has

$$E_1^{p,q} \simeq \hom_{S_p}(\mathbf{x}(p), \mathscr{H}^{p,q}).$$

To illustrate this, we observe that the key point of Chapter IV, which is the case in which $\mathbf{h} = \mathbf{e}$, is that we may take $\langle \mathscr{H}; q\rangle$ to be a sphere S^{q-2}, and $\mathscr{H}^{p,q} = 0$ for $q \neq 0$, while $\mathscr{H}^{p,0}$ is the sign representation $k[p]$ of S_p. In the general case, one must understand the various modules $\mathscr{H}^{p,q}$, hopefully via a geometric construction.

We have preliminary results for this in the case $\mathbf{h}$ is the species of linear orders. With the aid of a computer, we obtained the rank of $\mathscr{H}^{p,q}$ for $0 \leqslant p \leqslant 5$, which we list in Figure 1. The attentive reader might notice this table is nothing else than that of the *unsigned Stirling numbers of the first kind*. It would be desirable to understand the various S_p-modules $\mathscr{H}^{p,*}$ as completely as possible, as we did in the case of $\mathbf{e}$. Remark that we were able to prove $\mathscr{H}^{p,0}$ is the sign representation of S_p for every $p \in \mathbb{N}$.

Singular cohomology of a topological space. In Chapter III we assigned to every simplicial complex K a species whose cohomology is that of the suspension of K, see Proposition 5.2. It should be possible to do better, and assign to every topological space $\mathbf{x}$ a species whose cohomology algebra is that of $\mathbf{x}$. If possible, this would provide with a converse to the construction proposed in the beginning of this chapter.

A plausible idea is the following. First, one can show that to every cubical k-module Q one can assign a species $\mathsf{S}Q$ whose cohomology algebra coincides with that of Q;

[1] Read "$\mathbf{h}$ evaluated at q".

1	1	1	1	1	1
0	0	1	3	6	10
0	0	0	2	11	35
0	0	0	0	6	50
0	0	0	0	0	24
0	0	0	0	0	0

FIGURE 1. The Betti numbers for L.

in fact, we already know how to do this. Second, we can use the cubical set $Q(\mathbf{x})$ of singular cubes that Serre attaches in [**Ser1951**] to a topological space $\mathbf{x}$: it is a theorem this cubical complex calculates the ordinary cohomology algebra of such space. With this at hand the species $\mathsf{S}Q(\mathbf{x})$ does what we need.

Cohomology and classical operations on species. The category of species Sp_k is endowed, along with its monoidal Cauchy structure, with a wealth of operations that describe how combinatorial structures may be constructed from others, as thoroughly explained in [**LBL1998**]. It is desirable to understand if such operations restrict to the category of $\mathbf{e}$-bicomodules, that is, if $\mathbf{x}$ and $\mathbf{y}$ are $\mathbf{e}$-bicomodules, and if we write $\mathbf{x} * \mathbf{y}$ for the species obtained by applying a certain operation $? * ?$ to $\mathbf{x}$ and $\mathbf{y}$, does $\mathbf{x} * \mathbf{y}$ carry a "natural" bicomodule structure arising from that of $\mathbf{x}$ and $\mathbf{y}$? In that case, how is this reflected in the cohomology algebra of $\mathbf{x} * \mathbf{y}$? To illustrate, if $\mathbf{x}$ and $\mathbf{y}$ are $\mathbf{e}$-bicomodules, the species $\mathbf{x} \otimes \mathbf{y}$ is also an $\mathbf{e}$-bicomodule, and there is an arrow

$$H^*(\mathbf{x}) \otimes H^*(\mathbf{y}) \longrightarrow H^*(\mathbf{x} \otimes \mathbf{y}), \tag{11}$$

which may fail to be an isomorphism. Our calculations show this is an isomorphism, for example, when $\mathbf{x} = \mathbf{e}^{\otimes k}$ and $\mathbf{y} = \mathbf{e}^{\otimes n}$ for natural numbers n, k. This is, of course, an instance of a Kunneth-like theorem for the product $\otimes$ and the functor $C^*(?)$. As a second example, the species of Tr_* of rooted trees satisfies the following equation

$$\mathsf{Tr}_* = s \otimes (\mathbf{e} \circ \mathsf{Tr}_*),$$

where $\mathbf{e}$ is the exponential species, s is the species of singletons, and $\circ$ is a certain operation we did not define in this thesis. This equation has served wonders for enumerative purposes, and it would be remarkable if it aided, too, in the computation of the cohomology algebra of Tr_*.

Finally, let us record here we can now easily prove that for every pair of $\mathbf{e}$-bicomodules $\mathbf{x}$ and $\mathbf{y}$ there is a natural map

$$CC^*(\mathbf{x}) \otimes CC^*(\mathbf{y}) \longrightarrow C^*(\mathbf{x} \otimes \mathbf{y})$$

that is an isomorphism under mild hypothesis on one of $\mathbf{x}$ and $\mathbf{y}$, or if k is a field of characteristic zero. In this case, we obtain the desired result that the arrow (11) is an isomorphism. This has the favourable consequence that the cohomology algebra of a connected bimonoid in species is, in fact, a Hopf algebra, which should significantly aid in computations.

APPENDIX A

Homological algebra

We collect in the form of an appendix some results and constructions we used in Chapter IV. For further details, the reader can consult the texts mentioned towards the end of the Introduction.

1. Bifunctors and homology

Let $T : \mathsf{A} \times \mathsf{A} \longrightarrow \mathsf{B}$ be a bifunctor between abelian categories, covariant in the first variable and contravariant in the second, and consider arbitrary short exact sequences

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$
$$0 \longrightarrow C' \longrightarrow C \longrightarrow C'' \longrightarrow 0$$

in A. We say T is *left exact* if the sequences

$$0 \longrightarrow T(A'',C) \longrightarrow T(A,C) \longrightarrow T(A',C)$$
$$0 \longrightarrow T(A,C') \longrightarrow T(A,C) \longrightarrow T(A,C'')$$

are exact. This is true, for example, if T is an hom bifunctor. Dually, one defines *right exactness* of T, and then T is *exact* if it is both left and right exact. For the remaining of the section, let A and C denote chain complexes in A, and let $T(A,C)$ denote the standard double complex in B. The following results are the formal statements of the informal statement that "exact functors commute with homology". If A or C is concentrated in a degree and T is an hom bifunctor, this specializes to the commuting of homology and hom functors for projective or injective objects, which we used in Proposition 2.2.

PROPOSITION 1.1. [**CE1956**, Chapter IV, Proposition 6.1] *If T is right exact, there exists a homomorphism of degree zero $\alpha : H(T(A,C)) \longrightarrow T(H(A),H(C))$, natural in A and C.* ◀

PROPOSITION 1.2. [**CE1956**, Chapter IV, Proposition 6.1a] *If T is left exact, there exists a homomorphism of degree zero $\alpha' : T(H(A), H(C)) \longrightarrow H(T(A, C))$, natural in A and C.* ◀

It is important to remark these statements are incomplete: both α and α' are the identity of A and C have the trivial differential, and they make certain diagrams commutative, and this characterizes them uniquely.

THEOREM A.1.3. [**CE1956**, Chapter IV, Theorem 7.2] *If T is exact, α and α' are isomorphisms, and inverses of each other.* ◀

2. Spectral sequences

Fix an abelian category C, which usually is, for us, a category of representations of a ring. A *filtration* of a complex C in C consists of a family of subcomplexes $F = \{F^pC\}$ indexed by the integers and inclusions $F^{p+1}C \hookrightarrow F^pC$ for each integer p. We assume that $\{F^pC\}$ is *exhaustive*, meaning that $\bigcup F^pC = C$. We say that F is *bounded below* if for every n there is p such that $F^pC^n = 0$. Every filtration $\{F^pC\}$ of C induces a filtration on $H(C)$ so that

$$F^pH(C) = \mathrm{image}(H(F^pC) \longrightarrow H(C))$$

and an *associated bigraded object* $E_0(C)$ so that

$$E_0^{pq}(C) = F^pC^{p+q}/F^{p+1}C^{p+q}.$$

In particular, there is an associated bigraded object $E_0(H(C))$ associated to the filtration $\{F^pH(C)\}$.

A *cohomology spectral sequence (starting at r_0)* over C is a sequence (E, d) of bigraded objects with bigraded maps $\{(E_r, d_r)\}_{r \geqslant r_0}$ satisfying the following properties:

- Every d_r is a differential on E_r,
- The map d_r has bidegree $(r, 1-r)$,
- There are isomorphisms $H(E_r, d_r) \to E_{r+1}$ identifying $H(d_r)$ with d_{r+1}.

We will write E_r^{pq} for a generic bigraded component of E_r. A *homology spectral sequence* is defined dually by considering the homological grading $E^r_{pq} = E_{-r}^{-p,-q}$ and $d^r = d_{-r}$. Thus d^r has bidegree $(-r, r-1)$. There is a category of cohomology spectral sequences with morphisms $f : (E, d) \longrightarrow (E', d')$ those bigraded maps that preserve d and such that $H(f_r) \simeq f_{r+1}$ under the isomorphisms $H(E^r) \simeq E^{r+1}$.

Because each E^{r+1} is a subquotient of E^r, we can inductively define objects Z^r, B^r so that

$$B^r \subseteq B^{r+1} \subseteq \cdots \subseteq Z^{r+1} \subseteq Z^r$$

We set $Z^\infty = \cap Z^r$ and $B^\infty = \cup B^\infty$. Remark these objects always exist when dealing with complexes of abelian groups, but may fail to exist in other abelian categories. We may assume, however, that axioms **AB4** and **AB4*** are satisfied, and consider the appropriate limit and colimit objects.

A filtration is said to be *regular* if for each n we have $H^n(F^pC) = 0$ for p large enough depending on n. This implies, in particular, that $Z^r_{pq} = Z^\infty_{pq}$ for r large enough —in this case, we say that the spectral sequence is *regular*. If $\{F^pC\}$ is bounded below then the associated spectral sequence is regular, for bounded below filtrations are themselves regular. A spectral sequence is *weakly convergent* to a graded object H if there is a filtration $\{F^pH\}$ on H and an isomorphism $E_\infty \simeq E_0(H)$. We say a spectral sequence *converges* to a graded object H if it is regular, weakly converges to H and if the maps

$$u^n : H^n(C) \longrightarrow \varprojlim H^n(C)/F^pH^n(C)$$

are isomorphisms for each n. This last condition is immediate if the filtration $\{F^pC\}$ is bounded below. The following is the simplest of many theorems that ascertain the existence and convergence of a spectral sequence associated to a chain complex.

THEOREM A.2.1. *Every filtered complex C naturally determines a cohomology spectral sequence (E, d) starting at*

$$E_1^{pq} = H(F^pC^{p+q}/F^{p+1}C^{p+q}).$$

If the filtration is bounded below and exhaustive, this spectral sequence converges to $H(C)$. In particular, there are isomorphisms

$$E_\infty^{pq} \simeq F^pH^{p+q}(C)/F^{p+1}H^{p+q}(C).$$

In general, it is virtually impossible to compute a spectral sequence in its totality, as explained in [McC2001]:

"*It is worth repeating the caveat about the differentials mentioned in Chapter 1:* knowledge of E_r and d_r determines E_{r+1} but not d_{r+1}. *If we think of a spectral sequence as a black box with input a differential bigraded module, usually E_1, then with each turn of the handle, the machine computes a successive homology according to a sequence of differential. If some differential is unknown, then some other (*any other!*) principle is needed to proceed.*"

However, the mere existence of a spectral sequence allows for computation in favourable cases. The following are concrete examples of this last vague claim. We used a more refined variant of the second example in Chapter IV, Section 5.

EXAMPLE 2.2. A spectral sequence (E,d) *collapses at a page* $r \geqslant 2$ if there is either exactly one nonzero column in E^r or exactly one nonzero row in E^r. If (E,d) converges to H, then $H_n \simeq E_r^{pq}$ for the unique pair (p,q) with $p+q=n$.

EXAMPLE 2.3. Suppose C and C' are complexes with filtrations that are bounded below and exhaustive. If $f : C \longrightarrow C'$ is a morphism of filtered complexes there is an induced morphism of spectral sequences $f : E \longrightarrow E'$. If f_r is an isomorphism at some page E_r, one can show every f_s with $s \geqslant r$ is an isomorphism. This implies that $f_\infty : E_\infty \longrightarrow E'_\infty$ is an isomorphism. Because E_∞ and E'_∞ are the associated graded objects of $H(C)$ and $H(C')$, we can conclude f is a quasi-isomorphism.

3. Inverse limits and the Mittag-Leffler condition

We follow [Wei1994, Chapter 3, §5]. In classical abelian categories filtered colimits are exact functors, yet this is not the case for filtered limits. We can remedy this by use of the first derived functor of $\varprojlim$. This has a simple description when the chosen abelian category satisfies Grothendieck's axiom **AB4***: we ask that products of epimorphisms are epimorphisms, or put differently, that the product of exact sequences be exact. This is certainly true if the chosen abelian category is Ab, ${}_R\mathrm{mod}$ or $\mathrm{Ch}({}_R\mathrm{mod})$ where R is a ring.

A *tower of objects* C in an abelian category C is a diagram of the form

$$\cdots \longrightarrow C_{i+1} \longrightarrow C_i \longrightarrow \cdots \longrightarrow C_1 \longrightarrow C_0$$

which we denote by $\{C_i\}$. Put in other terms, a tower over C is an object in Ab^I where I is the poset

$$\cdots \longrightarrow i+1 \longrightarrow i \longrightarrow \cdots \longrightarrow 1 \longrightarrow 0$$

If $\{A_i\}$ is a tower in Ab (or a module category, or the category of chain complexes over a module category) there is a map

$$\Delta : \prod A_i \longrightarrow \prod A_i$$

$$(a_i)_i \mapsto (a_i - \overline{a_{i+1}})_i$$

where $\overline{a_{i+1}}$ is the image of a_{i+1} under the map $A_{i+1} \longrightarrow A_i$. It is clear that $\ker\Delta = \varprojlim A_i$. We define $\operatorname{coker}\Delta = \varprojlim^1 A_i$, and set $\varprojlim^n A_i = 0$ if $n \notin \{0,1\}$.

LEMMA 3.1. *Assume* C *is a complete abelian category with enough injectives and* J *is a small category. Then* C^J *has enough injectives. In particular, for* $J = I$*, there are enough injective objects of* C^I *that are products of towers of the form*

$$\cdots = E = E = E \longrightarrow 0 \longrightarrow \cdots$$

where E *is injective in* C*.*

LEMMA 3.2. *If all the maps in a tower* $\{A_i\}$ *are onto, then* $\varprojlim^1 A_i = 0$*.*

PROOF. We show that Δ is onto. Pick $(a_i) \in \prod A_i$, and any $b_0 \in A_0$, and let a_1 be a lift of $a_0 - b_0$ to A_1. Inductively, let a_{i+1} be a lift of $a_i - b_i$ to A_{i+1}. Then (b_i) covers (a_i) under Δ, and the claim follows. ◀

PROPOSITION 3.3. *The sequence of functors* $\{\varprojlim^n\}$ *form a* δ*-cohomological functor, and they are, in fact, the derived functors of the limit functor* $\varprojlim : \mathsf{Ab}^I \longrightarrow \mathsf{Ab}$*.*

PROOF. If $0 \longrightarrow \{A_i\} \longrightarrow \{B_i\} \longrightarrow \{C_i\} \longrightarrow 0$ is exact in Ab^I then axiom **AB4*** gives a diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \prod A_i \arrow[r] \arrow[d, "\Delta"] & \prod B_i \arrow[r] \arrow[d, "\Delta"] & \prod C_i \arrow[r] \arrow[d, "\Delta"] & 0 \\
0 \arrow[r] & \prod A_i \arrow[r] & \prod B_i \arrow[r] & \prod C_i \arrow[r] & 0
\end{tikzcd}$$

The snake lemma now gives the desired functorial exact sequence

$$
\begin{tikzcd}
0 \arrow[r] & \varprojlim A_i \arrow[r] & \varprojlim B_i \arrow[r] & \varprojlim C_i \arrow[dll, rounded corners] & \\
& \varprojlim^1 A_i \arrow[r] & \varprojlim^1 B_i \arrow[r] & \varprojlim^1 C_i \arrow[r] & 0
\end{tikzcd}
$$

To show that $\varprojlim^n \simeq R^n \varprojlim$ for each non-negative n, it suffices we show that $\{\varprojlim^n\}$ form a universal δ-functor. To prove this last claim, it suffices we show that $\varprojlim^1$ vanishes on enough injectives, the remaining $\varprojlim^n$ are already identically zero. By Lemma 3.1, there are enough injective objects $\{E_i\}$ in Ab^I that are a product of towers whose maps are onto, so the maps in $\{E_i\}$ are also onto by axiom AB4*. By Lemma 3.2 it follows that

$$\varprojlim^1 E_i = 0$$

which is what we wanted. ◀

DEFINITION 3.4. A tower of abelian groups $\{C_i\}$ satisfies the *Mittag-Leffler condition* if for every non-negative integer k there is $j \geqslant k$ such that, for every $i \geqslant j$, the image of $C_i \to C_k$ equals the image of $C_j \to C_k$. This is in particular true if $C_j \to C_k$ is the zero map.

PROPOSITION 3.5. *If a tower of abelian groups $\{C_j\}$ satisfies the Mittag-Leffler condition, then $\varprojlim^1 C_j = 0$.*

PROOF. Suppose first that $\{C_j\}$ satisfies the trivial Mittag-Leffler condition, and let us show that Δ is onto. Given $b_i \in C_i$ set $a_k = b_k + \overline{b_{k+1}} + \cdots + \overline{b_{j-1}}$ where j is such that the map $C_j \longrightarrow C_k$ is zero. Then (a_i) covers (b_i), as desired. Consider now an arbitrary tower $\{C_j\}$, and define a tower $\{A_j\}$ so that $A_j \subseteq C_j$ is the eventually stable image of the maps $C_i \longrightarrow C_j$. The maps in the tower $\{A_j\}$ are all onto so by Lemma 3.2 $\varprojlim^1 A_j = 0$, and the tower $\{C_j/A_j\}$ satisfies the trivial Mittag-Leffler condition, so $\varprojlim^1 C_j/A_j = 0$. Finally, from the short exact sequence of towers

$$0 \longrightarrow \{A_j\} \longrightarrow \{C_j\} \longrightarrow \{C_j/A_j\} \longrightarrow 0$$

we conclude that $\varprojlim^1 C_j = 0$, as claimed. ◀

The following theorem, necessary for the study of the spectral sequence in Chapter IV, is analogous to many results that precisely describe the failure of a left exact functor to preserve the homology of a complex when only its first derived functor is nonvanishing.

THEOREM A.3.6. *If $\{C_j\}$ is a tower of cochain complexes of abelian groups satisfying the Mittag-Leffler condition, there is a short exact sequence*

$$0 \longrightarrow \varprojlim{}^1 H^*(C_i)[1] \longrightarrow H^*(\varprojlim C_i) \longrightarrow \varprojlim H^*(C_i) \longrightarrow 0$$

PROOF. Write C for the complex $\varprojlim C_i$ and view the cycles Z and boundaries B of C as subcomplexes with zero differential. In particular, $H(C) = Z/B[-1]$ is also a complex with zero differential. Because $\varprojlim$ is left exact, $\varprojlim Z_i$ is exactly Z. Moreover, the exact sequence of towers of complexes

$$0 \longrightarrow \{Z_i\} \longrightarrow \{C_i\} \xrightarrow{d} \{B_i[-1]\} \longrightarrow 0$$

shows that $\varprojlim^1 B^i = 0$ and that the following sequence is exact

$$0 \longrightarrow B[-1] \longrightarrow \varprojlim B_i[-1] \longrightarrow \varprojlim Z_i \longrightarrow 0.$$

Similarly, from the exact sequence

$$0 \longrightarrow \{B_i\} \longrightarrow \{Z_i\} \longrightarrow \{H^*(C_i)\} \longrightarrow 0$$

it is deduced that $\varprojlim^1 Z_i$ and $\varprojlim^1 H^*(C_i)$ are isomorphic, and that

$$0 \longrightarrow \varprojlim B_i \longrightarrow Z \longrightarrow \varprojlim H^*(C_i) \longrightarrow 0.$$

is exact. All this shows that C is filtered by subcomplexes

$$0 \subseteq B \subseteq \varprojlim B_i \subseteq Z \subseteq C$$

whose filtration quotients are $B, \varprojlim^1 H^*(C_i)[1], \varprojlim H_*(C_i)$ and C/Z. This gives a short exact sequence

$$0 \longrightarrow \varprojlim B_i/B \longrightarrow Z/B \longrightarrow Z/\varprojlim B_i \longrightarrow 0$$

which, as just observed, defines a short exact sequence

$$0 \longrightarrow \varprojlim^1 H^*(C_i)[1] \longrightarrow H^*(C) \xrightarrow{\eta'} \varprojlim H^*(C_i) \longrightarrow 0,$$

and it is not hard to check η' is the canonical map $\eta : H^*(C) \longrightarrow \varprojlim H^*(C_i)$. This completes the proof of the theorem. ◀

Bibliography

[AM2006] Marcelo Aguiar and Swapneel Mahajan, *Coxeter groups and Hopf algebras*, Fields Institute Monographs, vol. 23, American Mathematical Society, Providence, RI, 2006. With a foreword by Nantel Bergeron. MR2225808 ↑79

[AM2010] ———, *Monoidal Functors, Species and Hopf Algebras*, American Mathematical Society and Centre de Recherches Mathématiques, 2010. ↑1, 24, 25, 27, 35, 41, 43

[AM2013] ———, *Hopf monoids in the category of species*, Hopf algebras and tensor categories, Contemp. Math., vol. 585, Amer. Math. Soc., Providence, RI, 2013, pp. 17–124. MR3077234 ↑24, 26, 35

[Bro1989] Kenneth S. Brown, *Buildings*, Springer-Verlag, New York, 1989. MR969123 ↑79

[CE1956] Henri Cartan and Samuel Eilenberg, *Homological algebra*, Princeton University Press, Princeton, N. J., 1956. MR0077480 ↑39, 40, 84, 107, 108

[Cop2015] Javier Coppola, *Cohomología en especies*, MSc Thesis, UdelaR, Facultad de Ingeniería - PEDECIBA, 2015. Available at `http://www.cmat.edu.uy/cmat/biblioteca/documentos/tesis/maestria/Cop2015`. ↑3

[Fre1964] Peter Freyd, *Abelian categories. An introduction to the theory of functors*, Harper's Series in Modern Mathematics, Harper & Row, Publishers, New York, 1964. MR0166240 ↑3

[Hat2002] Allen Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002. MR1867354 ↑84

[Joy1981] André Joyal, *Une théorie combinatoire des séries formelles*, Adv. in Math. **42** (1981), no. 1, 1–82. MR633783 ↑1, 3, 5

[Kas1995] Christian Kassel, *Quantum groups*, Graduate Texts in Mathematics, vol. 155, Springer-Verlag, New York, 1995. MR1321145 ↑3, 15, 19

[LBL1998] Gilbert Labelle, François Bergeron, and Pierre Leroux, *Combinatorial Species and Tree-like Structures*, Encyclopedia of Mathematics and its Applications, vol. 67, Cambridge University Press, 1998. Translated by Margaret Readdy. ↑3, 8, 104

[Lod1998] Jean-Louis Loday, *Cyclic homology*, 2nd ed., Grundlehren der Mathematischen Wissenschaften, vol. 301, Springer-Verlag, Berlin, 1998. Appendix E by María O. Ronco; Chapter 13 by the author in collaboration with Teimuraz Pirashvili. MR1600246 ↑43

[LQ1984] Jean-Louis Loday and Daniel Quillen, *Cyclic homology and the Lie algebra homology of matrices*, Comment. Math. Helv. **59** (1984), no. 4, 569–591. MR780077 ↑

[Mac1971] Saunders MacLane, *Categories for the working mathematician*, Springer-Verlag, New York-Berlin, 1971. Graduate Texts in Mathematics, Vol. 5. MR0354798 ↑3

[McC2001] John McCleary, *A user's guide to spectral sequences*, 2nd ed., Cambridge Studies in Advanced Mathematics, vol. 58, Cambridge University Press, Cambridge, 2001. MR1793722 ↑3, 76, 84, 109

[McK2017] Brendan McKay, *Graphs* (2017), `http://users.cecs.anu.edu.au/~ddm/data/graphs.html`. ↑96

[Ser1951] Jean-Pierre Serre, *Homologie singulière des espaces fibrés. Applications*, Ann. of Math. (2) **54** (1951), 425–505. MR0045386 ↑103

[Slo2017] N.J.A. Sloane, *The On-Line Encyclopedia of Integer Sequences* (2017), `http://oeis.org`. ↑95

[Sta1997] Richard P. Stanley, *Enumerative combinatorics. Vol. 1*, Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, Cambridge, 1997. With a foreword by Gian-Carlo Rota; Corrected reprint of the 1986 original. MR1442260 ↑

[Wei1994] Charles A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994. MR1269324 ↑3, 29, 31, 84, 102, 110