Algebraic operads (Part II)

Pedro Tamaroff
Trinity College Dublin

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What are operads *useful for*?

1. Organizing "higher structures" that extend usual products (Massey products, higher associativity)
2. Constructing nice adjunctions between categories of algebras (Quillen adjunction between commutative and Lie algebras, self-adjunction between associative algebras)
3. Doing algebra in the "third level" in the sense of Hinich (Gröbner–Shirshov bases, resolutions, syzygies)

What *are* operads?

An ns operad $\mathcal{P}$ is a sequence of operations $\underline{\mathcal{P}} = (\mathcal{P}(0), \mathcal{P}(1), \mathcal{P}(2), \ldots)$ along with a composition law

$$\gamma_{\mathcal{P}}^{e} : \mathcal{P}(T) \longrightarrow \mathcal{P}(T/e)$$

for each planar tree T and each edge $e \in E(T)$, that is "associative" in a generalized sense.

(Last time we only defined it for T a corolla. The axioms we gave show this more "combinatorial" picture is equivalent.)

What *are* operads?

An symmetric operad $\mathcal{P}$ is a sequence of operations with symmetric group actions $\underline{\mathcal{P}} = (\mathcal{P}(0), \mathcal{P}(1), \mathcal{P}(2), \ldots)$ along with a composition law

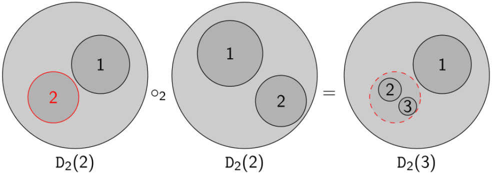
$$\gamma_{\mathcal{P}}^{e} : \mathcal{P}(T) \longrightarrow \mathcal{P}(T/e)$$

for each planar tree T and each edge $e \in E(T)$, that is "associative" in a generalized sense and compatible with the symmetric group actions.

Some examples

Some geometric operads

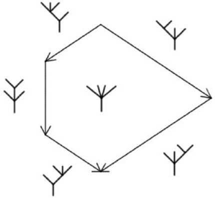
1. The operad of Stasheff $\mathcal{K}$ with $\mathcal{K}(n)$ the nth Stasheff polytope.
2. The little k-discs operad $\mathcal{C}_k$ with $\mathcal{C}_k(n)$ the space of affine embeddings of n disjoint k-discs into D^k.
3. The endomorphism operad of a topological space X with $\mathrm{End}_X(n) = \mathrm{Map}(X^n, X)$.



Algebraic examples

The three graces of Loday

1. The associative operad Ass with $\mathrm{Ass}(n) = \Bbbk S_n$.
2. The commutative operad Com with $\mathrm{Com}(n) = \Bbbk$.
3. The Lie operad Lie with $\mathrm{Lie}(n) = {}^{S_n}_{C_n}\!\uparrow \Bbbk_\xi$.



More exotic examples

More examples

1. The preLie operad preLie with $\mathrm{preLie}(n) = \Bbbk \mathrm{RT}(n)$.
2. The word operad $\mathbb{W}_M$ of a monoid M with $\mathbb{W}_M(n) = M^n$.
3. The braces operad Br with $\mathrm{Br}(n)$ the span of planar rooted trees with n vertices labelled by $[n]$ and some unlabelled vertices.

Tree monomials

In the same way a good understanding of monomials (words in some alphabet) is essential to work with associative algebras, a good understanding of tree monomials is essential to work with operads.

Running convention: all trees are rooted. The leaves are the only non-root vertices of degree one. The internal vertices are the vertices of degree at least two.

Definition

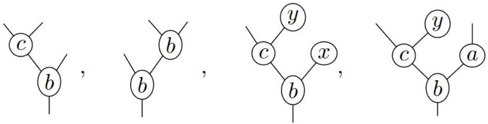
A ns tree monomial on $\mathcal{X}$ is a planar tree T with each of its internal vertices v decorated by an element of $\mathcal{X}(d)$ where $d = \mathrm{indeg}(v)$.

We write $\mathsf{Tree}_{\mathcal{X}}(n)$ the collection of ns tree monomials T on $\mathcal{X}$ with exactly n leaves, which we call the arity of T.

Example 3.3.2.3. Suppose that

$$\mathcal{X}(0) = \{x, y\}, \quad \mathcal{X}(1) = \{a\}, \quad \mathcal{X}(2) = \{b, c\}.$$

The following are examples of tree monomials in $\mathrm{Tree}_{\mathcal{X}}$:



The first two of them have arity 3 and weight 2, the third one has arity 1 and weight 4, and the last one has arity 2 and weight 4.

The free operad

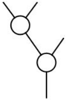
In the same way the free algebra on an alphabet V is obtained by taking the vector space spanned by monomials in V under the concatenation product:

Definition

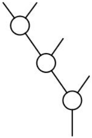
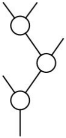
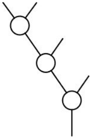
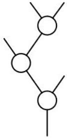
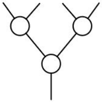
The free ns operad on an alphabet $\mathcal{X}$ is obtained by taking the symmetric collection $\mathcal{F}(\mathcal{X})$ where $\mathcal{F}(\mathcal{X})(n)$ is spanned by ns tree monomials in $\mathcal{X}$ of arity n. The ith partial composition product is obtained by grafting the root of a ns tree monomial onto the ith leaf of another ns tree monomial.

Examples

Consider the trees

$\tau_1 =$  $\tau_2 =$ 

Then we have the following compositions

$\tau_1 \circ_1 \tau_2$	$\tau_1 \circ_2 \tau_2$	$\tau_2 \circ_1 \tau_1$	$\tau_2 \circ_2 \tau_1$	$\tau_2 \circ_3 \tau_1$
				

Presentations

Once we have a free functor, we can talk about presentations.

An ideal in an operad $\mathcal{P}$ is a subsequence $\mathcal{I}$ stable under compositions when at least one argument belongs to $\mathcal{I}$.

An presentation of a ns operad $\mathcal{P}$ amounts to writing it as $\mathcal{F}(\mathcal{X})/(\mathcal{R})$ where $(\mathcal{R})$ is the ideal generated by some sequence $\mathcal{R}$ of relations.

Quadratic operads

An ns operad $\mathcal{P}$ is *quadratic* if it admits a presentation $\mathcal{F}(\mathcal{X})/(\mathcal{R})$ where $(\mathcal{R})$ is the ideal generated by some sequence $\mathcal{R}$ of relations in $\mathcal{F}(\mathcal{X})^{(2)}$.

For example, let $\mathcal{X}$ consist of a single binary operation $\mu = x_1x_2$, and consider the quadratic relation

$$x_1(x_2x_3) = (x_1x_2)x_3.$$

This gives a presentation of the ns associative operad.

We can "play the same game" in case we want to consider operations with symmetries. For example, let $\mathcal{Y}$ consist of a single binary operation $\beta = [x_1, x_2]$ so that $(12)\beta = -\beta$.

Then we can present Lie by $\mathcal{F}(\mathcal{Y})/(J)$ where J is the Jacobi identity:

$$J := (1 + \tau + \tau^2)\beta$$

where $\tau = (123)$. The only catch is that now ideals of relations are more complicated: we have to account for relations obtained from symmetric group actions, too!

Duality

There is an assignment $\mathcal{P} \longrightarrow \mathcal{P}^!$ for any symmetric quadratic operad $\mathcal{P}$. One can define it homologically using a "bar construction".

Basic idea: if $\mathcal{X} = \mathcal{X}(2)$, then $V = \mathcal{F}(X)^{(2)}$ is generated as an S_3-module by three kinds of compositions:

$$\mu(x_1, \nu(x_2, x_3)), \quad \mu(\nu(x_1, x_2), x_3), \quad \mu(\nu(x_1, x_3), x_2)$$

and we can define an S_3-equivariant pairing $V \otimes V^* \longrightarrow \mathbb{k}^-$ to define the orthogonal set of relations $R^\perp$.

The toolkit

1. Monomials and polynomials.
2. Rewriting rules.[1]
3. Leading monomials.
4. Division and overlappings.
5. Long division algorithm.
6. Normal forms and Gröbner–Shirshov bases.
7. Buchberger algorithm to compute GS bases.

[1] For example, coming from a monomial order.

The ns case

We already know what tree monomials and polynomials are.

A tree monomial order on $\mathcal{F}(\mathcal{X})$ is the data of total orders on each arity component so that composition is increasing in all of its arguments.

Example. Suppose we order $\mathcal{X}$ in some way. Then we can order monomials in $\mathcal{X}$. We can use *path sequences* to order tree monomials.

Path sequences

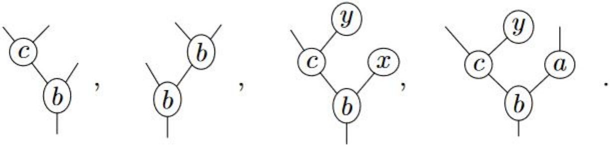
The path sequence of a ns tree monomial T is a tuple of monomials in $\mathcal{X}^*$ obtained by “reading T from top to bottom”.

The main feature of path sequence is they store ns tree monomials faithfully. So we can use orders on usual monomials in free monoids to order ns tree monomials.

Example 3.4.1.3. Suppose that

$$\mathcal{X}(0) = \{x, y\}, \quad \mathcal{X}(1) = \{a\}, \quad \mathcal{X}(2) = \{b, c\}.$$

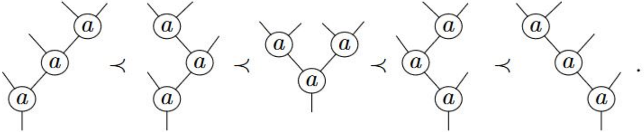
Let us consider the tree monomials from Example 3.3.2.3



The corresponding path sequences are, respectively,

$$(bc, bc, b), \quad (b, bb, bb), \quad (bc, bcy, bx), \quad (bc, bcy, ba).$$

Example 3.4.1.8. Let $\mathcal{X}_2 = \{a\}$. For the `gpathlex` order, we have



This follows from comparing the corresponding path sequences

$$(a, a^2, a^3, a^3) \prec (a, a^3, a^3, a^2) \prec (a^2, a^2, a^2, a^2) \prec$$
$$\prec (a^2, a^3, a^3, a) \prec (a^3, a^3, a^2, a).$$

Divisibility and overlappings

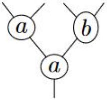
What does it mean for a ns tree monomial T to divide another ns tree monomial T'?

Combinatorially, T divides T' if we can find T as a subtree of T' with the correct labels.

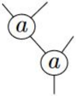
Algebraically, T divides T' if we can obtain T' from T by composing it in some way with other monomials.

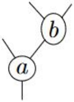
These two notions coincide! Because operadic composition is much richer than algebraic concatenation, this notion of divisibility is more complicated.

Example 3.4.2.5. Let $\mathcal{X} = \{a, b\}$. The monomial

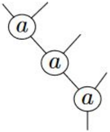


$\in \mathcal{T}(\mathcal{X})(4)$

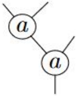
has two different divisors of weight 2: the "left divisor" 

and the "right divisor" 

. In comparison, the tree monomial



$\in \mathcal{T}(\mathcal{X})(4)$

has two divisors which are both occurrences of the monomial 

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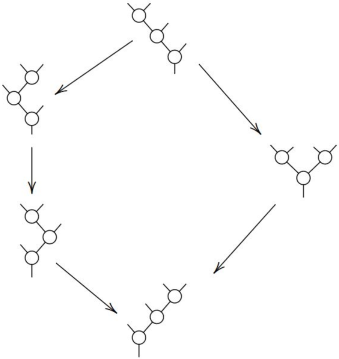
Once we have a notion of leading monomials, division and overlappings, we can do rewriting in ns operads.

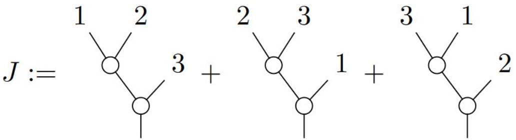
We have a long division algorithm, a notion of Gröbner basis, and a notion of normal forms, along with a Diamond Lemma.



What about symmetries?

It is sometimes difficult or even impossible (!) to work with objects that are too symmetric.

For example, let us consider Lie with the presentation $\mathcal{F}(\beta)/(J)$ where β is the bracket and J the Jacobi identity:

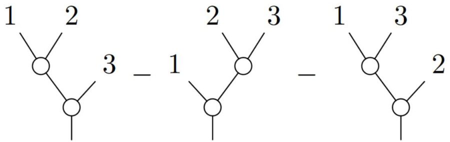


Which one is the leading term?

There is no way to choose a leading term: any of the three terms is in the ideal generated by the other, in fact, in the same S_3-orbit!

How is this solved? One uses shuffle operads, that "forget about symmetries" in a delicate way.

In the previous case, we can instead consider J written as follows:



A theory of overlappings

What happens with the homological side of rewriting theory?

Can we produce a nice theory in the lines of Anick's paper *On the Homology of Associative Algebras*? (Transactions of the American Mathematical Society Vol. **296**, No. 2)

Up to an extent, things seem to work fine!