Rational and differentiably finite power series

Notes for the Undergraduate Combinatorics Seminar

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1 Reminders

1.1 Binomial series

(1.1.1) We will always denote a sequence of numbers with a lower case latin letter, and its respective ordinary generating function with the corresponding capital latter. Thus, for example, the generating function of a sequence $f : \mathbb{N}_0 \longrightarrow \mathbb{C}$, which we usually denote by (f_n) for brevity, will be denoted by F.

(1.1.2) For $c \in \mathbb{C}$, define the *binomial series with parameter c* by the power series

$$\beta_c(z) = \sum_{n \geqslant 0} \binom{c}{n} z^n, \text{ where } \binom{c}{n} = \frac{c(c-1)\cdots(c-n+1)}{n!}.$$

It can be shown that this series converges in the open ball $B(0,1)$ and coincides there with $(1+z)^c = \exp(c\log(z))$ where log denotes the principal branch of the complex logarithm.

(1.1.3) We can use the above to obtain a power series expansion for $(1-\gamma z)^{-d}$, where $d \in \mathbb{N}$ and $\gamma \in \mathbb{C}$ is a fixed scalar: we have

$$\frac{1}{(1-\gamma z)^d} = \sum_{n \geqslant 0} \binom{-d}{n} (-1)^n \gamma^n z^n.$$

Now a direct computation shows that for any $d, n \in \mathbb{N}$ we have

$$\binom{-d}{n} = (-1)^n \binom{n+d-1}{d-1}$$

which gives us the desired representation:

$$\frac{1}{(1-\gamma z)^d} = \sum_{n\geqslant 0} \binom{n+d-1}{d-1} \gamma^n z^n.$$

1.2 Finite differences

(1.2.1) In what follows, it will be useful to have in mind that the $\mathbb{C}$-vector space $\mathbb{C}[n]$ of polynomials in n with complex coefficients has basis $\mathscr{B}_0 = \{(n)_0, (n)_1, (n)_2, \cdots\}$, where $(n)_j = n(n-1)\cdots(n-j+1)$. We call $(n)_j$ a *falling factorial power* or *binomial polynomial*. By translation, it follows that for any $i \in \mathbb{Z}$, the family $\mathscr{B}_i = \{(n+i)_0, (n+i)_1, (n+i)_2, \cdots\}$ is also a basis.

(1.2.2) In fact, much like the usual basis $1, n, n^2, n^3, \ldots$ is related to the iterated derivatives of a polynomial, in the sense that the coefficient of n^j in a polynomial P is given by $P^{(j)}(0)/j!$, the new basis $\mathscr{B}_0$ is related to the iterated finite differences —what can be thought as a discrete analog of differentiation— in such a way that the coefficient of $(n)_j$ in a polynomial P is given by $\Delta^j P(0)/j!$, where in general for $n \in \mathbb{N}_0$,

$$\Delta^j P(n) = \sum_{u=0}^{j} (-1)^{j-u} \binom{j}{u} P(n+u)$$

is the jth finite difference of P at $n \in \mathbb{N}_0$. For a gentle introduction to finite differences, the reader can consult [1].

2 Rational functions

2.1 A classical example

(2.1.1) The problem of enumerating a sequence of sets $\{X_0, X_1, \ldots\}$ can sometimes be tackled by obtaining a recursive formula for $\#X_{n+1}$ in terms of $\#X_0, \#X_1, \ldots, \#X_n$. The best case scenario happens when the recursion happens to be linear. Let us begin with an example.

(2.1.2) For each $n \in \mathbb{N}$, let f_n denote the numbers of ways of writing n as an ordered sum of elements of $\{1,2\}$, and let F_n denote the collection of such ordered tuples. In particular $f_0 = 0$ and $f_1 = 1$, since there is no way of writing zero as a sum of elements of $\{1,2\}$, while we can write 1 exactly in one way as a sum of elements of such set.

(2.1.3) Assume now that $n \geqslant 2$, and let us show that there is a bijection between F_n and the union $F_{n-1} \cup F_{n-2}$. This will certainly prove that for every $n \geqslant 2$, we have

$$f_n = f_{n-1} + f_{n-2}. \tag{1}$$

Concretely, given $t = (x_1, \dots, x_\ell) \in F_n$, consider $t' = (x_1, \dots, x_{\ell-1})$, that is, delete the last term. If $x_\ell = 2$ then $t' \in F_{n-2}$, and if $x_\ell = 1$ then $t' \in F_{n-1}$. We have thus defined a map

$$\phi : t \in F_n \longmapsto t' \in F_{n-1} \cup F_{n-2},$$

whose inverse is readily described: if $s \in F_{n-1}$, append a 1 to the end of s, and if $s \in F_{n-2}$, append a 2 to the end of s.

(2.1.4) We now exploit the above recurrence to obtain the ordinary generating function of the sequence (f_n). Multiplying (1) by z^n and summing through $n \geqslant 2$ we obtain that

$$F(z) - z = F(z)(z + z^2) \tag{2}$$

or, what is the same, that F is a rational function of z:

$$F(z) = \frac{z}{1 - z - z^2}.$$

It is perhaps not yet evident why this is useful. The point is we can now factor $1 - z - z^2$ as $(1 - \psi z)(1 - \bar{\psi} z)$ where $\psi = \frac{1+\sqrt{5}}{2}$ and $\bar{\psi} = 1 - \psi$ and use partial fractions to deduce that

$$F(z) = \frac{z}{\sqrt{5}} \left(\frac{1}{1 - \psi z} - \frac{1}{1 - \bar{\psi} z} \right).$$

Using the geometric series now provides us with a closed formula for the nth term of (f_n):

$$f_n = \frac{1}{\sqrt{5}} (\psi^n - \bar{\psi}^n). \tag{3}$$

(2.1.5) It turns out that (1), (2) and (3) are all equivalent statements. However, each one has its own use depending on one's objective. The recursion (1) is useful to deduce certain surprising identities as, for example, the following *Catalan identity*: for any $n, r \in \mathbb{N}$ we have

$$f_n^2 - f_{n+r} f_{n-r} = (-1)^{n-r} f_r^2.$$

On the other hand, (3) is certainly useful to first estimate the growth of (f_n), and second, to compute f_n in a non-recursive way: since we have that $\bar{\psi}^n/\sqrt{5} < 1/2$ for $n \in \mathbb{N}$, it follows that

$$f_n = \left[\frac{\phi^n}{\sqrt{5}}\right]$$

for any $n \geqslant 0$ where $[-]$ denotes the nearest integer function. The reader can try to find situations in which (2) is more useful than the other two results, since I have not come up with one myself.

2.2 The main result

We now state and prove a theorem which shows that the results just obtained for the sequence (f_n) are part of a general phenomenon.

Theorem 2.1. *Fix $d \in \mathbb{N}$ and $\alpha_1, \dots, \alpha_d \in \mathbb{C}$ with α_d nonzero, and let $Q(z) = 1+\alpha_1 z+\cdots+\alpha_d z^d$. Then, for a function $f : \mathbb{N}_0 \longrightarrow \mathbb{C}$, the following statements are equivalent:*

(1) *The generating function F is rational, of the form P/Q where P is a polynomial of degree less than d.*

(2) *For every $n \geqslant 0$, we have the recursion $\alpha_d f_n + \alpha_{d-1} f_{n-1} + \cdots + \alpha_1 f_{n+d-1} + f_{n+d} = 0$.*

(3) *For every $n \geqslant 0$, we have the formula $f_n = \sum_{i=1}^k P_i(n)\gamma_i^n$, where $Q(z) = \prod_{i=1}^k (1-\gamma_i z)^{d_i}$ and each P_i is a polynomial of degree less than d_i.*

Proof. Define V_i for $i \in \{1,2,3\}$ to be the subspace of functions $f : \mathbb{N}_0 \longrightarrow \mathbb{C}$ such that the ith condition holds, and define V_4 to be the subspace of functions $f : \mathbb{N}_0 \longrightarrow \mathbb{C}$ whose generating function is of the form

$$F(z) = \sum_{i=1}^{k} q_i(z)(1-\gamma_i z)^{-d_i}$$

where for each $i \in \{1, \dots, k\}$, q_i is a polynomial of degree less than d_i.

We observe that for $i \in \{1,2,3,4\}$ we have $\dim_{\mathbb{C}} V_i = d$. Indeed, V_1 has dimension d since the space of polynomials of degree less than d has this dimension, V_2 has dimension d because the linear recurrence for (f_n) determines f uniquely in terms of $f_0, \dots, f_{d-1}$, V_3 has dimension d since the d coefficients of the polynomials $P_1, \dots, P_k$ determine (f_n) uniquely, and finally V_4 has dimension d by this last argument on the polynomials $q_1, \dots, q_k$.

We now deduce that $V_1 \subseteq V_2$ by equating coefficients in $QF = P$, that $V_4 \subseteq V_1$ by taking a common denominator in the expression of F, and finally we argue that $V_4 \subseteq V_1$. Indeed, by linearity, it suffices we notice that for each $c, j \in \mathbb{N}_0$ and $\gamma \in \mathbb{C}$ we have

$$\frac{z^j}{(1-\gamma z)^c} = \sum_{n\geqslant 0} \binom{n+c-1-j}{c-1} \gamma^{n-j} z^n$$

where $\binom{n+c-1-j}{c-1}\gamma^{-j}$ is a polynomial of degree $c-1$. Since comparable vector spaces of equal dimension are in fact equal, we deduce that $V_1 = V_2 = V_3 = V_4$, which is what we wanted. ◂

(2.2.1) This theorem characterizes only those power series F that are rational functions of the form P/Q where P has degree less than Q, but it easy to deduce a general statement from it. Indeed, if F is rational of the above form but $\deg P \geqslant \deg Q$, we can use long division to write $F = R + S/Q$ with S, R polynomials and with $\deg S < \deg Q$. It follows by looking at $F - R$ that for large values of $n > t$ where $t = \deg R$, the coefficients of F satisfy a linear recurrence as in the last theorem. Conversely, if for large values of n the coefficients of F satisfy a linear recurrence, the above theorem yields that $F = R + S/Q$ as before.

(2.2.2) It is obvious that if F and G are power series that are rational, so is their usual Cauchy product FG. We can define another product $F \star G$, known as the *Hadamard product*, so that the nth coefficient of $F \star G$ is $f_n g_n$. A remarkable corollary of the theorem we have proven is that

Corollary 2.2. *The Hadamard product of two rational power series is again rational.*

Proof. Indeed, using the third characterization of rational power series given by the theorem, we know that for n large we have closed formulas

$$f_n = \sum P_i(n)\gamma_i^n \text{ and } g_n = \sum Q_i(n)\delta_i^n$$

where each P_i and each Q_i is a polynomial. We then have that for large enough n,

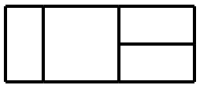
$$f_n g_n = \sum P_i(n) Q_j(n) (\gamma_i \delta_j)^n,$$

which, by another use of the theorem and the discussion in (2.2.1), shows $F \star G$ is itself a rational function. ◂

2.3 Exercises

The following exercises are intended to get a grasp of the contents of the section.

(1) Let f_n denote the number of ways of tiling a $2 \times n$ board with 2×2 and 1×2 tiles, where the 1×2 tile is allowed to be positioned both vertically or horizontally. The following shows a valid tiling for the 2×5 case.



Obtain a linear recursion for the sequence (f_n) and hence exhibit F as a rational function, and give a closed formula for its coefficients.

(2) Prove the following precise form of the discussion made in (2.2.1).

Theorem 2.3. *Let $f : \mathbb{N}_0 \longrightarrow \mathbb{C}$, and suppose that $F = P/Q$ where P and Q are polynomials. Then there is a unique finite set $E_f \subseteq \mathbb{N}_0$, called the* exceptional set of f*, and a unique function $f_1 : \mathbb{N}_0 \longrightarrow \mathbb{C}^\times$ such that the function $g : \mathbb{N}_0 \longrightarrow \mathbb{C}$ defined by*

$$g(n) = \begin{cases} f(n) & \text{if } n \notin E_f, \\ f(n) + f_1(n) & \text{if } n \in E_f. \end{cases}$$

satisfies $G = S/Q$ where S is a polynomial with $\deg S < \deg Q$*. Moreover, if E_f is nonempty, then the largest element m in it is* $\deg P - \deg Q$ *–or, equivalently, the degree of the polynomial R in our previous discussion– and m is also equal both to the largest integer for which statements* (2) *and* (3) *in Theorem 2.1 fail to hold.*

(3) Deduce from what we have proven that the following three statements are equivalent for a coefficient function $f : \mathbb{N}_0 \longrightarrow \mathbb{C}$:

(1) We have $F = P/(1-z)^{d+1}$ where P is a polynomial of degree at most d,
(2) For all $n \geqslant 0$ we have $\Delta^{d+1} f(n) = 0$, that is, $\Delta^{d+1} f = 0$.
(3) The function f is a polynomial function in n of degreemat most d.

Recall we defined Δ in (1.2.2). This shows that, much like the case of ordinary differentiation, the polynomial coefficient functions are those that eventually have zero discrete derivative.

3 D-finite and P-recursive functions

3.1 Differentiably finite functions

(3.1.1) Write $\mathbb{C}[\![z]\!][z^{-1}]$ for the set of formal Laurent series with finite principal part, that is, expressions of the form

$$\sum_{n\geqslant n_0} a_n z^n$$

where $n_0 \in \mathbb{Z}$. This is a vector space over the field of rational functions $\mathbb{C}(z)$ by usual multiplication of power series. We will say a power series F is differentiably finite, or D-finite, for short, if the $\mathbb{C}(z)$-span of F and all its derivatives is finite dimensional in $\mathbb{C}[\![z]\!][z^{-1}]$. For brevity, we will use ∂ to denote the derivative with respect to the variable z. The following lemma sheds some light on our last definition.

Lemma 3.1. *The following are equivalent for a powerseries $f \in \mathbb{C}[\![z]\!]$:*

(1) *The series F is D-finite,*

(2) *There exist polynomials $q_0, \ldots, q_k, q$, not all zero such that*

$$\sum_{t=0}^{k} q_t \partial^t F = q. \tag{4}$$

(3) *There exist polynomials $p_0, \ldots, p_m$ not all zero such that*

$$\sum_{t=0}^{m} p_t \partial^t F = 0. \tag{5}$$

(3.1.2) Observe that if we introduce the differential operators with polynomial coefficients

$$D = \sum_{t=0}^{k} q_t \partial^t, D' = \sum_{t=0}^{m} p_t \partial^t$$

then (4) says that $DF \in \mathbb{C}[z]$, while (5) says that $D'F = 0$. This shows, in particular, that D-finite power series are a $\mathbb{C}$-linear subspace of $\mathbb{C}[\![z]\!][z^{-1}]$. But much more is true, as we will soon prove. Note, also, that rational power series are manifestly D-finite in light of (4), for if $F = P/Q$ then taking the operator $D = Q$ gives $DF \in \mathbb{C}[z]$.

Proof. Suppose that F is D-finite, and that the linear span of F and all its derivatives has dimension d. Then there exist a linear dependence relation among the $d+1$ power series

$F, \partial F, \ldots, \partial^d F$, and clearing denominators in this relation gives an equation (4) with $q = 0$, so (1) $\Longrightarrow$ (2). If such an equation holds, and if q has degree t, then differentiating this equation $t+1$ times gives an equation of the form (5), so that (2) $\Longrightarrow$ (3). Finally, consider an equation (5) where $p_m \neq 0$. Dividing by p_m now gives that $\partial^m F$ is in the rational span of $f, \partial f, \ldots, \partial^{m-1} F$. Differentiating (5) and repeating the argument now shows $\partial^{m+1} F$ is in the rational span of $f, \partial f, \ldots, \partial^{m-1} F$. Inductively, it follows that the linear span of F and all its derivatives has dimension at most m, so (3) $\Longrightarrow$ (1). ◀

(3.1.3) Observe that D-finite functions are an extension of rational functions in the sense we allow for differentiation in a linear dependence equation, and not only the appearance of polynomials. Because differentiating a power series is equivalent to multiplying its coefficients by the falling power polynomials introduced in (1.2.1), that is,

$$\partial^t F = \sum_{n \geqslant t} (n)_t f_n z^{n-t},$$

the following definition should not come off as a surprise.

3.2 Polynomially recursive functions

(3.2.1) We say a function $f : \mathbb{N}_0 \longrightarrow \mathbb{C}$ is polynomially recursive, or P-recursive, for short, if there exist $d \in \mathbb{N}_0$ and polynomials $P_0, \ldots, P_d$ with $P_d \neq 0$ such that for all $n \geqslant 0$,

$$P_d(n) f_{n+d} + \cdots + P_0(n) f_n = 0 \tag{6}$$

(3.2.2) It is immediate, as opposed to the rational case, that the property of f being P-recursive is not affected by modifying it in a finite subset S of $\mathbb{N}_0$: we can always multiply (6) by a polynomial Q having roots in S, and we now obtain an equation that shows the modified version of f is still P-recursive. Note that the coefficients functions of a rational power series is, again, manifestly P-recursive, for our result shows we can take $P_d, \ldots, P_0$ to be constant.

(3.2.3) It seems valuable to observe that we can give a linear algebraic definition of P-recursive functions that mimics that for D-finite power series. Let us say two coefficient functions $f, g : \mathbb{N}_0 \longrightarrow \mathbb{C}$ have the same *germ* if they agree for large values of n. This defines an equivalence relation on the set of all such functions, and we write $[f]$ for the equivalence class of f and call it the *germ of f*. By the above, if f and g have the same germ, then f is P-recursive if and only if g is, so we can talk about P-recursive germs. The following lemma is our desired analogue. We define $(Sf)(n) = f(n+1)$, so that in general $(S^i f)(n) = f(n+i)$.

Lemma 3.2. *A function $f : \mathbb{N}_0 \longrightarrow \mathbb{C}$ is P-recursive if and only if the span of the germ of f and its shifts $\{[f], [Sf], [S^2 f], \ldots\}$ is a finite dimensional subspace of the space of all germs, when viewed as a $\mathbb{C}(n)$-vector space.*

Proof. Luckily, this is just a fancy restatement of P-recursiveness, so the proof is easy, but it will come in handy in what follows. If f is P-recursive, then we can write by (6)

$$[S^d f] = \sum_{i=0}^{d-1} \frac{P_i(n)}{P_d(n)} [S^i f]$$

so that $S^d f$ is in the $\mathbb{C}(n)$-span of $[f], [Sf], \ldots, [S^{d-1} f]$. Now shifting gives all higher shifts of $[f]$ are also in this span. Conversely, if the $\mathbb{C}(n)$-span of $[f]$ and its shifts is finite dimensional, clearing denominators we obtain an equation (6) for the germ of f, and we can deduce f is P-recursive for large values of n, so it must be P-recursive. ◀

3.3 The equivalence of both definitions

We can now prove the desired analogue of Theorem 2.1 for D-finite functions.

Theorem 3.3. *A powerseries F is D-finite if and only if f is P-recursive.*

Proof. Suppose that F is D-finite and recall, once again, that for any $t, s \in \mathbb{N}_0$ we have

$$z^t \partial^s F = \sum_{n \geqslant t} (n+s-t)_s f_{n+s-t} z^n$$

Since $(n+s-t)_s$ is a polynomial in n, equating coefficients in (5) gives a recurrence of the form (6). Conversely, suppose that f is P-recursive, so we have a recursion of the form (6). Recall that we can write $P_i(n)$ in the basis $\mathscr{B}_i$ of binomial polynomials introduced in (1.2.1), and we can now note that

$$\sum_{n \geqslant 0} (n+i)_j f_{n+i} z^n = z^{j-i} \partial^i F + R$$

where R is a possibly zero polynomial in z and z^{-1}: note that it may happen that $j < i$, so some finitely many terms in the right hand side have to be fixed to match up the left hand side. It follows now that multiplying (6) by z^n and summing, we will obtain an equation of the form (4) after clearing some negative powers of z. ◀

(3.3.1) At this point it would be valuable to provide examples of D-finite power series that are not rational. We were previously introduced to the generating function of the Catalan numbers, which can be defined by $c_0 = 0$ and by the recursion

$$c_{n+1} = \sum_{t=1}^{n} c_t c_{n+1-t}$$

for $n \geqslant 0$. We have shown that

$$C(z) = \frac{1-\sqrt{1-4z}}{2},$$

which shows C is D-finite. Indeed, one way to see this is that it suffices we show $S = \sqrt{1-4z}$ is D-finite, but this satisfies the equation $2PS' - P'S = 0$ where $P = 1-4z$. More generally, if R is rational and if $k \in \mathbb{N}_0$, any kth root S of R is D-finite although generally not rational, for it satisfies the equation $kRS' - R'S = 0$ from where we can clear denominators to obtain an equation of the form (5).

(3.3.2) Yet another way to note that Catalan numbers are P-recursive is to exhibit a P-recursion for them, and since we know that

$$c_{n+1} = \frac{1}{n+1}\binom{2n}{n}$$

we can now note that $(n+1)c_{n+1} - 2(2n-1)c_n = 0$ for every $n \geqslant 1$. We can also use the following corollary of Theorem 3.3.

Theorem 3.4. *The Cauchy and Hadamard product of D-finite powerseries is again D-finite.*

Proof. Both statements are proved with the same technique. If you know about tensor products of vector spaces, the idea is that if V and V' are finite dimensional and we want to show a third space W is finite dimensional, we can do so by producing a linear transformation $T : V \otimes V' \longrightarrow V''$ so that $\operatorname{im}(T)$ contains W. It then follows that

$$\dim W \leqslant \dim \operatorname{im}(T) \leqslant (\dim V)(\dim V') < \infty$$

so W is indeed finite dimensional.

So suppose that F and G are both D-finite, and let us show that the Cauchy product FG is D-finite, too. It suffices we show that FG and their derivatives span a finite dimensional $\mathbb{C}(z)$-vector space. Now note that if $\{F, \partial F, \ldots, \partial^t F\}$ and $\{G, \partial G, \ldots, \partial^s G\}$ are bases for the span of F and its derivatives and the span of G and its derivatives, respectively, then, by Leibniz's rule, the span of the set $\{\partial^i F \partial^j G : i \in [\![1,t]\!], j \in [\![1,s]\!]\}$ contains the span of FG and its derivatives, so FG is also D-finite. To show that $F \star G$ is D-finite, we instead show its coefficients are

P-recursive. To do so, we use Lemma 3.2, and show that the span of the germs of $[fg]$ and its shifts is finite dimensional. Again, we note that since $[f]$ and $[g]$ and their shifts span a finite dimensional subspace of the space of all germs, the span of $\{[S^i f S^j g] : i, j \in \mathbb{N}_0\}$ is finite dimensional, and it contains the span of $\{[S^i f S^i g] : i \in \mathbb{N}_0\}$ so $[fg]$ is P-recursive, and so is fg, as we wanted. ◄

References

[1] Richard P. Stanley, *Enumerative combinatorics. Vol. 1*, Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, Cambridge, 1997. With a foreword by Gian-Carlo Rota; Corrected reprint of the 1986 original. MR1442260

[2] R. P. Stanley, *Differentiably finite power series*, European J. Combin. **1** (1980), no. 2, 175–188. MR587530