An introduction to cohomology

Pedro Tamaroff

21st December 2018

Contents

1 Motivation

1.1 What can cohomology do for you?

(1.1.1) Let us consider the following problems: can we determine the rational points in the unit sphere S^1, that is, pairs (x, y) of rational numbers such that $x^2 + y^2 = 1$? Given a compact manifold X and a self-map $X \to X$, can we determine if it has fixed points? If so, how many are there? Also, can we be determine the least number of contractible open subsets one can cover it with? Can we determine, for each $n \in \mathbb{N}$, how many linearly independent vector fields there are in the n-dimensional sphere? Given a set in Euclidean space defined by a finite number of polynomials, can we cut it out by less polynomials? Can we determine how many lines intersect four given lines in general position in projective 3-space?

(1.1.2) It turns out that all of these problems can be solved, in one way or another, through the judicious use of a cohomology theory. The following cohomology theory correspond to the problems above, in the given order: Galois cohomology, singular and de Rham cohomology, operations on topological K-theory, local cohomology of ideals in rings, and finally Chow rings and intersection theory, specifically, Schubert calculus. The extent to which one can say these problems are entirely solved by cohomological methods varies from problem to problem and, of course, there are cases when cohomology is not the most natural approach: one can easily determine rational points in the sphere with basic geometry! But this just illustrates cohomology does not escape the mundane.

1.2 Cohomology: experiments, measurements and data storage

(1.2.1) There are at least three ways to think about cohomological invariants of objects: (i) they are experiments we perform on complicated objects that we cannot understand, so we probe them, (ii) they are measurements we perform on objects, to distinguish them, classify them and even determine them completely, or up to certain equivalences we are comfortable with and, (iii) they are gadgets that store data from an object in a very refined fashion: this data could be stored in a different way, but this would be useless for our purposes. To illustrate, de Rham cohomology fits all three descriptions, many theories for algebras fit the description of (ii), while the second Galois cohomology group of a field extension is an example of (iii).

(1.2.2) A fourth way cohomology theories arise is as corrections to our wishful thinking: in many situations, we are studying an assignment from objects to, for example, vector spaces, and we would like this assignment to respect certain structure in our objects. This is usually partially true, and cohomology appears naturally as a way to produce a correct statement replacing our wishful thinking. For example, one can study bundles on topological spaces, but assigning a bundle to its global sections does not respect extensions of bundles, and cohomology fixes this problem.

(1.2.3) Another of the achievements of cohomology theory is endowing the stored data with extra structure: associative algebra structures, Lie algebra structures and cohomology operations, among others. This replaces our objects of interest with a very rich algebraic gadget, and it turns out that these enhanced invariants have played a fundamental role in solving open questions and in studying the objects that give rise to them to a remarkable level of detail.

2 First examples and definitions

2.1 Cocycles in nature: deformations of algebras

(2.1.1) Let us consider the following problem: suppose given an algebra A over the complex or real numbers, and let $A[\varepsilon]$ be the space of infinitesimal polynomials on A, where $\varepsilon^2 = 0$. We want to impose on $A[\varepsilon]/(\varepsilon^2)$ a new product $\star$, which we assume is bilinear for the new variable, so that we need only determine it on coefficients, say $a \star b = ab + f(a,b)\varepsilon$.

(2.1.2) We claim that this product is unital for the usual unit of $A[\varepsilon]/(\varepsilon^2)$ if and only if $f : A \times A \to A$ is zero whenever one argument is $1 \in A$, and that it is associative if and only if for any three $a, b, c \in A$, the following *cocycle condition* holds, in which case we call f a *cocycle*:

$$af(b,c) - f(ab,c) + f(a,bc) - f(a,b)c = 0.$$

Indeed, $a \star 1 = a + f(a,1)\varepsilon$, and since 1 and ε are an A-basis for $A[\varepsilon]$, we want that $f(a,1) = 0$ for any $a \in A$, and similarly $f(1,a) = 0$ for any $a \in A$. Let us now consider the associativity of $\star$. We can compute that

$$(a \star b) \star c = (ab)c + (f(a,b)c + f(ab,c))\varepsilon,$$
$$a \star (b \star c) = a(bc) + (af(b,c) + f(a,bc))\varepsilon,$$

so that $\star$ is associative if and only if f satisfies the cocycle condition.

(2.1.3) If we denote by $Z^2(A)$ the collection of cocycles $f : A \times A \to A$, then we have just shown we have a bijection between the set of *infinitesimal deformations of A to first order* and the set $Z^2(A)$. But it turns out that two different cocyles can give equivalent deformations: we say two deformations are equivalent if there is an automorphism $G : A[\varepsilon] \to A[\varepsilon]$ of the form $G(a) = a + g(a)\varepsilon$ that sends one product to the other. In such case, writing f_1 and f_2 for the cocycles giving rise to such deformations, we have that for $a, b \in A$,

$$G(a \star b) = G(ab + f_1(a,b)\varepsilon) = ab + g(ab)\varepsilon + f_1(a,b)\varepsilon.$$

On the other hand, we compute that for $a, b \in A$,

$$G(a) \star G(b) = (a + g(a)\varepsilon) \star (b + g(b)\varepsilon) = ab + (ag(b) + g(a)b + f_2(a,b))\varepsilon.$$

This means that two deformations are equivalent if and only if there is a self-map $g : A \to A$ such that $f_1(a,b) - f_2(a,b) = ag(b) - g(ab) + g(a)b$.

(2.1.4) This motivates the following defitnition: a cocycle is called trivial or a *coboundary* if it is of the form $f(a,b) = ag(b) - g(ab) + g(a)b$. In this case the associated deformation is equivalent to the trivial one where $a \star b = ab$. We write $B^2(A)$ for the space of such cocycles, and observe we have a obtained a bijection between the equivalence class of first order infinitesimal deformations of A and the quotient space $\mathrm{HH}^2(A) = Z^2(A)/B^2(A)$ of cocycles up to boundaries. This is called the *second Hochschild cohomology group of A*.

Exercise 1. Prove that every coboundary is indeed a cocycle, a fact we used without proof in the definition of $\mathrm{HH}^2(A)$ for A and algebra. That is, show that if $f : A \times A \to A$ is of the form $f(a,b) = ag(b) - g(ab) + g(a)b$ for some $g : A \to A$, then f satisfies the cocycle equation.

Exercise 2. Suppose that $f_1 : A \times A \to A$ and A_1 is the associated deformation of A to first order, and consider the space of $A[\varepsilon]/(\varepsilon^3)$ where now $\varepsilon^3 = 0$, but $\varepsilon^2 \neq 0$: this consists of polynomials $a_0 + a_1\varepsilon + a_2\varepsilon^2$ with multiplication given by the rule that ε^3 is zero. We consider now a product here where

$$a \star b = ab + f_1(a,b)\varepsilon + f_2(a,b)\varepsilon^2.$$

What extra conditions must f_1 and f_2 satisfy in order that this defines an associative product extending that of A_1? When such a map $f_2 : A \times A \to A$ exists, we say that f_1 is integrable to second order. Can you work out the conditions for integrability to an arbitrary fixed order? We point to reader to [4, 5] for more information on this.

Exercise 3. Consider the algebra of *dual numbers* $\mathbb{C}[t]/(t^2)$ and the Weyl algebra generated by two operators x, y with $xy = qyx$, where $q \in \mathbb{C}^\times$ is not a root of one. Show that the first algebra has $\mathrm{HH}^2(A)$ of dimension one, while the Weyl algebra has no non-trivial deformations of first order, that is, $\mathrm{HH}^2(A) = 0$ in this case. What is the explicit multiplication of the only non-trivial deformation of the algebra of dual numbers?

2.2 Chain complexes

(2.2.1) We now develop the necessary machinery to explain that the previous example falls into the realm of cohomology theories and homological algebra. Useful references for the general theory of homological algebra include [3, 10, 15].

(2.2.2) We begin by observing that if for $n \in \mathbb{N}_0$, we write $C^n(A)$ for the space of all functions $A \times \cdots \times A \to A$, then what we have been looking at is at a sequence of linear maps $C^1(A) \xrightarrow{d^1} C^2(A) \xrightarrow{d^2} C^3(A)$ where for $f \in C^2(A)$ and $g \in C^1(A)$, we have $(d^1 g)(a,b) = ag(b) - g(ab) + g(a)b$, and $(d^2 f)(a,b,c) = af(b,c) - f(ab,c) + f(a,bc) - f(a,b)c$. The cocycles $Z^2(A)$ are precisely the elements in the kernel of d^2, and the trivial ones, or coboundaries, are those in the image of d^1, and the fact every coboundary is a cocycle can be encoded in the equation $d^2 d^1 = 0$. Then we defined $\mathrm{HH}^2(A) = \ker d^2 / \operatorname{im} d^1$. This motivated Gerhard Hochschild [8], and many others before him, to consider the following objects.

(2.2.3) A *cochain complex* is a family of vector spaces indexed by the integers, say $C = \{C^n\}_{n \in \mathbb{Z}}$, along with *boundary maps* $d^n : C^n \longrightarrow C^{n+1}$ such that $d^{n+1} d^n = 0$ for each $n \in \mathbb{Z}$. We will say that an element in C^n has degree n, that an element is an n-cocycle if it is in the kernel of d^n. Elements in the image of d^{n-1} are called n-coboundaries, and we write $Z^n(C)$ and $B^n(C)$ for the spaces of n-cocycles and n-coboundaries, respectively. We will usually write $d^2 = 0$ whenever we want to say the family of maps $\{d^n : C^n \longrightarrow C^{n+1}\}$ satisfies $d^{n+1} d^n = 0$ for each $n \in \mathbb{Z}$.

(2.2.4) We now observe that, by definition, $B^n(C) \subseteq Z^n(C)$ for each $n \in \mathbb{Z}$, so we write $H^n(C)$ for the quotient $Z^n(C)/B^n(C)$, which we call the *nth cohomology group of C*. This is the space of equivalence classes $[z]$ of n-cocycles under the equivalence relation that $z \sim z'$ if $z - z' = d_{n-1}(c'') \in B^n(C)$ for some $c'' \in C^{n-1}$. Thus we identify two cocycles if their difference is a coboundary. If we do not want to worry about degrees, we will use the notation $Z(C)$, $B(C)$ and $H(C)$ for the sequence of spaces we just introduced.

(2.2.5) A map $f : C \to D$ between chain complexes is a family of linear maps $\{f^n : C^n \longrightarrow D^n\}_{n \in \mathbb{Z}}$ so that for each $n \in \mathbb{Z}$, $f^{n+1} d_C^n = d_D^n f^n$. We write this condition, more simply, by $df = fd$, and say that f and d commute. It means that f sends cocycles in C to cocycles in D, and coboundaries in C to coboundaries in D, so that we have an induced map $H(f) : H(C) \to H(D)$ so that $H(f)[z] = [f(z)]$. That is, $H(f)$ sends the equivalence class $[z]$ of a cocycle z to the equivalence class $[f(z)]$ of the cocycle $f(z)$.

(2.2.6) What we are *really* interested in is not in chain complexes as stand-alone gadgets, but rather in the following picture. Let us fix a class of objects C, which could consist of algebras, manifolds, topological spaces, symplectic manifolds, vector bundles over some space, or groups. To each object x in this class, we will assign a cochain complex of vector spaces, let us call it $C(x)$, in such a way that whenever we have a map between objects $f : x \longrightarrow y$, we have a corresponding map between complexes $C(f) : C(y) \longrightarrow C(x)$. This assignment should be compatible with our class C, in the sense that if f is a composition of maps, say $f = f_1 f_2$, then $C(f)$ is be the composition $C(f_2)C(f_1)$, and if f is the identity map of x, then $C(f)$ is the identity map of $C(x)$. Succinctly, we want a functor $C : \mathsf{C} \longrightarrow \mathsf{Ch}$ from some category C of interest to chain complexes.

(2.2.7) A fundamental concept of homological algebra is that of an exact sequence. We say a sequence of maps, which we depict as a diagram $V' \xrightarrow{f} V \xrightarrow{g} V''$ is *exact at* V if the image of f equals the kernel of g. Note that this means in particular that $gf = 0$, so this diagram is a cochain complex, and exactness at V means precisely that the homology $\ker g / \operatorname{im} f$ is zero: every cocycle is a coboundary. We will say a complex is exact if it has zero homology at every integer, and will then call such a complex a *long exact sequence*. A *short exact sequence* is one of the form $0 \longrightarrow V' \xrightarrow{f} V \xrightarrow{g} V'' \longrightarrow 0$; that this sequence be exact means that f is injective, g is onto, and $\ker g = \operatorname{im} f$.

(2.2.8) A *short exact sequence of cochain complexes* consists of a diagram of the form $0 \longrightarrow C' \xrightarrow{f} C \xrightarrow{g} C'' \longrightarrow 0$ where f and g are maps of complexes such that for each $n \in \mathbb{Z}$ the sequence $0 \longrightarrow C'^n \xrightarrow{f^n} C^n \xrightarrow{g^n} C''^n \longrightarrow 0$ is exact. It is not hard to see that in this case, for each $n \in \mathbb{Z}$, the sequence $H^n(C') \to H^n(C) \to H^n(C'')$ is exact. The cornerstone of homological algebra is the following result, which says that to each short exact sequence of chain complexes we can assign a long exact sequence of their cohomology groups by gluing these three terms sequences together. From this long exact sequence, we can then relate an extract information in a "two our of three principle": if we know enough information about the cohomology of two of our complexes, we expect to know enough about the cohomology of the third.

Theorem. *Suppose that we have a short exact sequence of complexes as before. For each $n \in \mathbb{Z}$ there is a linear map $\partial^n : H^n(C'') \longrightarrow H^{n+1}(C')$ so that the following sequence is exact: $H^{n-1}(C'') \xrightarrow{\partial^{n-1}} H^n(C') \xrightarrow{H(f)} H^n(C) \xrightarrow{H(g)} H^n(C'') \xrightarrow{\partial^n} H^{n+1}(C')$.*

We call the family of maps $\partial = \{\partial^n\}$ the *connecting morphisms* of the sequence.

Proof. Fix an integer $n \in \mathbb{Z}$. We define the *boundary map* $\partial^n : H^n(C'') \longrightarrow H^{n+1}(C')$ and leave to the reader to prove that the sequence above is exact. Let us then take an n-cocycle $[z'']$ in $H(C'')$, that is, a class represented by an element z such that $dz'' = 0$. Since the map $C \to C''$ is onto, we can find some $c \in C$ so that $g(c) = z''$. Because g and d commute, it follows that dc is in the kernel of g: $g(dc) = df(c) = dz = 0$, although it may not be a cocycle. Finally, because the sequence is exact, dc must be in the image of f, so there is some $z' \in C'$ such that $f(z') = dc$. Moreover, because f is injective, $dz' = 0$, since $f(dz') = df(z') = d^2c = 0$. We then set $\partial^n[z''] = [z']$, and note that $[z'] \in H^{n+1}(C')$. One has to check then that: (i) this is well defined, that is, the class $[z']$ depends only on the class $[z'']$ and not on the choices we made to define it, (ii) the image of ∂^{n-1} is the kernel of $H^n(f)$, (iii) the kernel of ∂^n is the image of $H^n(g)$. ◀

Exercise 4. We say a map of complexes $f : C \longrightarrow D$ is a quasi-isomorphism if the induced map $H(f) : H(C) \longrightarrow H(D)$ is an isomorphism. Show that quasi-isomorphisms satisfy the following property: if $g : D \to E$ is a second map, then whenever two out of three of f, g and gf are quasi-isomorphisms, so is the third. To do this, show that $H(gf) = H(g)H(f)$.

Exercise 5. Show that if $0 \to C' \to C \to C \to 0$ is an exact sequence of complexes, then the sequence $H(C') \to H(C) \to H(C'')$ is exact. Can you give an example where $H(C') \to H(C)$ is not injective and one where $H(C) \to H(C'')$ is not surjective in some degree $n \in \mathbb{Z}$? Can you give an example where both situations happen? Use the next exercise to produce counter-examples!

Exercise 6. Let M be a manifold, N a submanifold, and let $\Omega^*(M) \longrightarrow \Omega^*(N)$ be the restriction of forms. Show this map is surjective. The kernel of this is denoted $\Omega^*(M,N)$, and its homology is called the relative de Rham cohomology of the pair (M,N). Show the sequence of complexes $0 \to \Omega^*(M,N) \to \Omega^*(M) \to \Omega^*(N) \to 0$ is exact and that the connecting morphism $H^*(N) \longrightarrow H^{*+1}(M,N)$ can be described as follows: if ω is a closed form in N, let $\overline{\omega}$ be a form on M whose restriction is ω. Then $\partial[\omega] = [d\overline{\omega}]$. Note that $d\overline{\omega}$ is a cocycle but not necessarily zero.

Exercise 7. Let $\{U, V\}$ be an open cover of M by open subsets. Show that the sequence $0 \to \Omega^*(M) \to \Omega^*(U) \oplus \Omega^*(V) \to \Omega(U \cap V) \to 0$ is exact, where the first map is the restriction and the second sends (ω, ω') to the difference of their restrictions. The long exact sequence associated to this is called the Mayer–Vietoris sequence for the cover $\{U, V\}$. What happens for covers by more open sets?

3 The cohomology of spaces

3.1 Singular cochains

(3.1.1) We now introduce singular cohomology of topological spaces. In what follows, fix a topological space X, which we will usually want to think is a smooth manifold. For $n \in \mathbb{N}_0$, a *singular n-simplex* on X is a continuous map $\sigma : \Delta^n \to X$. In particular, a 0-simplex is just a point, and a 1-simplex is a path in X. A singular n-chain in X is a formal combination of singular n-simplices, $\lambda_1\sigma_1 + \cdots + \lambda_t\sigma_t$, where the coefficients are complex numbers. We write $C_*(X)$ for the vector space of such singular chains, and $C^*(X)$ for its dual. Elements in $C^*(X)$ are called singular cochains: these are functions which assign a complex number to each singular chain in $\mathbb{C}$. We proceed to make it into a complex of vector spaces.

(3.1.2) If $\sigma : \Delta^n \longrightarrow X$ is a singular simplex and if $i \in \{0, \dots, n\}$, the ith face of σ, which we write σ^i, is the map $\sigma_i : \Delta^{n-1} \longrightarrow X$ that is obtained by precomposing σ with the map that that embeds $\Delta^{n-1} \subset \Delta^n$ as those points with ith barycentric coordinate zero. Then define $d\sigma$ to be the singular chain $\sum_{i=0}^n (-1)^i \sigma^i$ obtained as the alternating sum of faces of σ. It is a pleasant exercise to check that $d : C_*(X) \to C_*(X)$ squares to zero, and then so does the map $d : C^*(X) \to C^*(X)$ such that $(d\varphi)(\sigma) = \varphi(d\sigma)$. In this way we obtain a cochain complex of singular cochains on X. We define the *singular cohomology of* X to be the homology of $C^*(X)$, and write it $H^*(X)$.

3.2 Relative cohomology and cup products

(3.2.1) Suppose now that $Y \subseteq X$ is a subspace. The inclusion map $i : Y \to X$ allows us to assign a singular cochain in X to one in Y by restriction: every singular simplex in Y is one in X, so we can evaluate any singular cochain in X at a simplex in Y. This defines a map $i^* : C^*(X) \to C^*(Y)$. The kernel of this map, that is, those singular cochains whose restriction to Y is zero, is denote by $C^*(X, Y)$ and is called the cochain complex of singular cochains in X relative to Y. Its cohomology is the cohomology of X relative to Y. Because the map $i^* : C^*(X) \to C^*(Y)$ is surjective, we have an exact sequences of complexes $0 \longrightarrow C^*(X, Y) \longrightarrow C^*(X) \longrightarrow C^*(Y) \longrightarrow 0$ and the long exact sequence associated to this is called the long exact sequence of relative cohomology for the inclusion $Y \subseteq X$.

(3.2.2) Suppose that σ is an n-simplex in X. We define its ith front face ${}_i\sigma$ to be the i-simplex obtained by restricting σ to its first i barycentric coordinates, and its ith back face to be the $(n-i)$-simplex σ_{n-i} obtained by restricting σ to its last $n-i$ barycentric coordinates. We now define a product of singular cochains as follows. Suppose that f is a p-cochain and g is a q-cochain, and that σ is a $(p+q)$-simplex. We define $f \smile g$ to be the $p+q$-cochain so that for each such simplex, $(f \smile g)(\sigma) = f({}_p\sigma)g(\sigma_q)$. It is straightforward to check that $\smile : C^*(X) \times C^*(X) \longrightarrow C^*(X)$ is an associative unital product that makes $C^*(X)$ into an algebra. Moreover, it is compatible with the differential on cochains in the following sense: if f and g are cochains, and g is in degree p, then we have the *Leibniz formula* $d(f \smile g) = df \smile g + (-1)^p f \smile dg$. From this it follows that if f and g are cocycles then so is their product, and if one of them is a boundary, so is their product, so that there is a well defined product in $H^*(X)$, which we also call the cup product.

(3.2.3) We now observe that if $A, B \subseteq X$ are subspaces, then $C^*(X,A) \smile C^*(X,B) \subseteq C^*(X, A \cup B)$, that is, the cup product of two cochains, one which vanishes on A and the other which vanishes in B, vanishes in the union $A \cup B$. This means that when considering relative cohomology groups, what we actually have is a map $\smile$: $H^*(X,A) \times H^*(X,B) \longrightarrow H^*(X, A\cup B)$. Of course, we can iterate this to any number of factors, and this observation will be crucial in the following subsection.

3.3 The cup length and Lusternik–Schnirlemann category

(3.3.1) A space X has *cup length at most n* if every n-fold product of possibly distinct cohomology classes in X vanish, and it has *cup length n* if it has cup length at most n but not at most $n-1$. We write $\mathrm{cl}(X)$ for this integer, which may be infinite. In a similar fashion, a space has LS category at most n if it can be covered by n open sets such that each inclusion $U \subseteq X$ is null-homotopic, and that is has LS category n if it has LS category at most n but not at most $n-1$, and write $\mathrm{cat}(X)$ for this number, which, again, way be infinite. We can now prove the following result.

Theorem. *If X is contractible, then $H^*(X) = 0$ for $* > 0$. More generally, let X be a space and suppose that we have n subspaces $U_1, \ldots, U_n$ whose union is X, and so that the inclusions $U_i \subseteq X$ are null-homotopic. Then any product of n cohomology classes in $H^*(X)$ is zero. In other words, we have that* $\mathrm{cl}(X) \leqslant \mathrm{cat}(X)$ *for every space X. In particular, if we can find $\xi \in H^*(X)$ such that $\xi^n \neq 0$, we cannot cover X by n open contractible sets.*

(3.3.2) We say that an inclusion $U \subseteq X$ is *null-homotopic* if U can be shrunk to a point in X, and that a space X is *contractible* if the inclusion $X \subseteq X$ is null-homotopic. For example, for every convex set U in Euclidean space X, the inclusion $U \subseteq X$ is nullhomotopic. On the other hand, the incusion of S^1 in $\mathbb{R}^2 \setminus (0,0)$ is not null-homotopic. One can check that if $U \subseteq X$ is null-homotopic, then $H^*(X) \to H^*(U)$ is the zero map when $* > 0$: we will use this in what follows.

Proof. To prove the first claim, note that if X is contractible, then the identity $X \to X$ is null-homotopic, and the map $H^*(X) \to H^*(X)$, which is also the identity, is zero. This means that $H^*(X)$ is the zero vector space, since this is the only map whose identity map is zero. Let us now put ourselves in the situation where X is covered by n subspaces whose inclusions are null-homotopic, and pick cocycles $\varphi_1, \ldots, \varphi_n$ representing cohomology classes in $H^*(X)$. We wish to show that $\varphi_1 \smile \cdots \smile \varphi_n$ is zero. To this end, let us begin by noting that for each $i \in \{1, \ldots, n\}$ we have an exact sequence $H^*(X, U_i) \longrightarrow H^*(X) \longrightarrow H^*(U_i)$. The last map is zero, so the statement is that the map $j^* : H^*(X, U_i) \longrightarrow H^*(X)$, which views a class of a cocycle which vanishes on U_i as a cocycle in X, is surjective. This means that for each $i \in \{1, \ldots, n\}$, we can find a cocycle ψ_i such that $j^*[\psi_i] = [\varphi_i]$. Moreover, by Exercise 9, if we let $\xi_i = [\psi_i]$ and $\xi'_i = [\varphi_i]$, we have that

$$j^*(\xi_1 \smile \cdots \smile \xi_n) = \xi'_1 \smile \cdots \smile \xi'_n.$$

It follows that, to show the product on the right vanishes, we can show the one on the left does. But this is obtained by a product map on relative cohomology groups

$$H^*(X, U_1) \times \cdots \times H^*(X, U_n) \longrightarrow H^*(X, U_1 \cup \cdots \cup U_n) = H^*(X, X)$$

which vanishes because $C^*(X, X) = 0$, and so $H^*(X, X) = 0$. ◀

Exercise 8. Although the wedge product on forms is super-commutative, the cup product operation on $C^*(X)$ is not super-commutative: if f is a cocycle in degree p and g is one in degree q, then $[f, g] = -(-1)^{p+q} d(f \smile_1 g)$ where $\smile_1$ is the 1-cup product of Steenrod. Can you find an expression for it, at least in low degrees? Higher cup products were defined by Steenrod in [12], they witness the non-commutativity of the cup product in a coherent way, and can be used to describe all *stable operations* on the cohomology of spaces.

Exercise 9. Suppose that $f : X \to Y$ is a map between spaces. Show that there is a map of cochain complexes $f^* : C^*(Y) \to C^*(X)$ so that if σ is a singular chain in X and if φ is a singular cochain in Y, then $f^*(\varphi)(\sigma) = \varphi(f\sigma)$. Show that f^* respects cup products in the sense that if ψ is another singular cochain in Y, then $f^*(\varphi \smile \psi) = f^*(\varphi) \smile f^*(\psi)$.

Exercise 10. Show that one can cover projective n-space by $n+1$ contractible open subsets, but not by n. In particular, deduce that one can cover the sphere by two contractible open sets, but that the sphere is not contractible. What is the cup lenght of the n-torus? And what is its Lusternik–Schnirelmann category? Can you find a space with infinite cup length?

4 Answers and stranger things

4.1 The motivating examples of the introduction

We now give a summary of the way cohomology theories give answers to the questions posed in the introduction.

1. Rational points on the sphere. To each field extension $F \subseteq E$ with Galois group G, one can consider the group cohomology $H^*(G, F^\times)$, which is called the Galois cohomology group of the extension. Hilbert's *Satz 90* says that the first cohomology group of a cyclic extension vanishes. We apply this to the extension $\mathbb{Q}(i)/\mathbb{Q}$, where we know how to compute $H^1(G, F^\times)$: this is simply the kernel of the norm map $1+\tau : \mathbb{Q}(i) \to \mathbb{Q}$ such that $a + ib \mapsto a^2 + b^2$ modulo the image of the map $1-\tau : \mathbb{Q}(i) \to \mathbb{Q}(i)$ such that $T(z) = z\bar{z}^{-1}$. Since $H^1(C_2, \mathbb{Q}^\times)$ is zero, every rational point on the sphere is of the form $z\bar{z}^{-1}$ for some $s+it$. Unwinding the definitions we see that any rational point on the sphere is of the form $((s^2-t^2)/(s^2+t^2), 2st/(s^2+t^2))$. where s, t can be chosen to be *natural numbers*. This gives us the usual parametrization of Pythagorean triples by $X = s^2 - t^2$, $Y = 2st$, and $Z = s^2 + t^2$: these are precisely the triples (X, Y, Z) of integers such that $X^2 + Y^2 = Z^2$.

2. Fixed points on manifolds. Given a self-map $f : X \to X$ on a compact manifold, we can consider the *Lefschetz number of f*, given by $L(f) = \sum_{i=0}^n (-1)^i \operatorname{Tr}(f^* : H^i(X) \longrightarrow H^i(X))$. The *Leftschetz fixed point theorem* says that if $L(f)$ is non-zero, then f has at least one fixed point. In fact, it shows that $L(f)$ is the sum over the fixed points of f weighed by the degree of f at that point. Because these can be negative, it may happen that $L(f) = 0$, even though f has fixed points. One can also consider *critical points* of self-maps, that is, points of some $f : X \longrightarrow X$ where $df_p = 0$, and the minimum number $F(X)$ of critical points a self map can have. It turns out that $\mathrm{cat}(X) \leqslant F(X)$, where the first number is the Lusternik–Schnirelmann category introduced in these notes. The book [7] contains the elements of fixed point theory.

3. Linearly independent vector fields on spheres. For a long time, it was known that there were at least $p(n) - 1$ linearly independent vector fields on the n-sphere, but it was not known whether this bound was sharp [1]. To any manifold one can assign a cohomology ring $K(X)$, called its K-*theory*. By introducing certian operations $\Phi : K(X) \longrightarrow K(X)$, and analyzing them in the case of real projective spaces and quotients of them by projective subspaces, Frank Adams proved that this bound is sharp. Other operations on cohomology rings played a fundamental role in algebraic topology, such as the Steenrod powers and Bockstein homomorphisms. Frank Adams introduced other operations, called *secondary operations*, that are defined via usual operations, which are now called *primary*. We refer the reader to [13].

4. Spaces cut by polynomial equations. Let A be a commutative ring and suppose I is an ideal in A. For each A-module M, we can define a submodule $\Gamma_I(M) = \{x \in M : I^i x = 0 \text{ for some } i \in \mathbb{N}_0\}$. This functor is left exact, and its right derived functor is called the *local cohomology of M with support on I* and denoted by $H_I^*(M)$. It turns out that if I can be generated by n elements, then $H_I^i(M) = 0$ for $i > n$, so that if we can find an A-module M such that $H_I^n(M) \neq 0$, we can conclude that I cannot be generated by less than n elements. If we take A to be a polynomial ring and I to be an ideal defined by (finitely many) polynomials, this method can be used to determine when algebraic sets cannot be cut by a certain amount of equations. Local cohomology enjoys very nice properties, such as the existence of a Mayer-Vietoris sequence involving the sum and intersection of ideals. This is just the manifestation on the algebraic side of the fact local cohomology can be defined, more generally, for sheaves on schemes relative to a closed subscheme. See [6, 15].

5. Lines intersecting lines. If X is a projective variety, then we can assign it its Chow ring $A_*(X)$, which consists of certain equivalence classes of subvarieties. In favourable situations, one can endow $A_*(X)$ with an intersection product that corresponds, geomtrically, to the intersection of subvarieties. If X is the variety of lines in projective 3-space or, what is the same, the Grassmann variety $G(2,4)$, one can show that $A_*(X)$ is generated by six classes: s_4 corresponds to $X \subseteq X$, s_3 to the lines intersecting a given line in $\mathbb{P}^3$, s_{21} to those passing through a given point, s_{22} to the lines contained in a plane, s_1 to those contained in a plane and passing through a point in this plane and, finally, s_0 to those passing through two given points. The product s_3^4 is an element that lives in $A_0(X)$, which one can prove is the set of integers $\mathbb{Z}$, and is equal to 2: there are two lines intersecting four other fixed lines in general position.

4.2 Beyond here lie Demogorgons

(4.2.1) Entire books have been written regarding cohomology theories, like K-theory, Galois cohomology, cohomology of schemes, local cohomology, equivariant cohomology theories, Čech cohomology, Hochschild cohomology, Chevalley–Eilenberg cohomology, ètale cohomology, and others, and it is impossible to give a full picture in such a short set of notes. It is worth remarking that cohomology theories are usually conceived as tools to solve problems, or at least this has been the mindset in modern times, the shining example of this being the idea of André Weil that a collection of conjectures, going back to ideas of Emil Artin, should be a consequence of the existence of a cohomology theory for varieties over finite fields, having certain properties similar to those of other cohomology theories. One of the outstanding achievements of algebraic geometers of the 20th century was the construction of Weil cohomology theories and the subsequence proof of the Weil conjectures, as envisioned by Weil. These involved the work of Bernard Dwork, Alexander Grothendieck, Pierre Deligne, Jean-Louis Verdier and Jean-Pierre Serre, among others. See [9].

(4.2.2) A rather striking state of affairs is that cohomology is *very hard to compute*, even though it can computed in a wide variety of ways. Early techniques developed to compute cohomology groups involved Mayer–Vietoris sequences and long exact sequences, for example. As cohomology theories became more intricate, these techniques evolved. A major advance was the use of spectral sequences by Jean Leray to compute cohomology of sheaves on topological spaces. He developed them in captivity after he was taken prisoner by the German forces in 1940 and sent to camp a in Austria, where he remained until the end of the war. Spectral sequences have now become an invaluable tool of computation, and there are ongoing collective efforts to study certain spectral sequences of interest, such as Frank Adams' spectral sequences [11, Chapter 9], which compute the stable homotopy groups of spheres. Other spectral sequences once relevant to algebraic topology are the Serre spectral sequence and the Eilenberg–Moore spectral sequences.

(4.2.3) Around the same time as Jean Leray, Henri Cartan and Samuel Eilenberg developed the general language of homological algebra, along with the seminal *Tohoku paper* of Alexander Grothendieck. All three developed the idea of doing cohomology theory in abelian categories, which includes the notion of resolutions, that of left and right exact functors and their derived functors, external and internal products in the resulting cohomology groups, and others. In this way, they showed how the gadgets developed by algebraic topologists, for example, were part of a theory of their own right, and explained how to obtain their results from a general theory.

References

[1] J. F. Adams, *Vector fields on spheres*, Ann. of Math. (2) **75** (1962), 603–632, DOI 10.2307/1970213. MR0139178

[2] Kenneth S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics, vol. 87, Springer-Verlag, New York-Berlin, 1982. MR672956

[3] Henri Cartan and Samuel Eilenberg, *Homological algebra*, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 1999. With an appendix by David A. Buchsbaum; Reprint of the 1956 original. MR1731415

[4] Murray Gerstenhaber, *On the deformation of rings and algebras*, Ann. of Math. (2) **79** (1964), 59–103, DOI 10.2307/1970484. MR0171807

[5] ———, *On the deformation of rings and algebras. II*, Ann. of Math. **84** (1966), 1–19, DOI 10.2307/1970528. MR0207793

[6] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52. MR0463157

[7] Allen Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002. MR1867354

[8] G. Hochschild, *On the cohomology groups of an associative algebra*, Ann. of Math. (2) **46** (1945), 58–67, DOI 10.2307/1969145. MR0011076

[9] Frans Oort, *The Weil conjectures*, NAW 5/15 **3** (2014), 211–219. Available at http://www.nieuwarchief.nl/serie5/pdf/naw5-2014-15-3-211.pdf.

[10] Saunders MacLane, *Homology*, 1st ed., Springer-Verlag, Berlin-New York, 1967. Die Grundlehren der mathematischen Wissenschaften, Band 114. MR0349792

[11] John McCleary, *A user's guide to spectral sequences*, 2nd ed., Cambridge Studies in Advanced Mathematics, vol. 58, Cambridge University Press, Cambridge, 2001. MR1793722

[12] N. E. Steenrod, *Products of cocycles and extensions of mappings*, Ann. of Math. (2) **48** (1947), 290–320, DOI 10.2307/1969172. MR0022071

[13] Robert E. Mosher and Martin C. Tangora, *Cohomology operations and applications in homotopy theory*, Harper & Row, Publishers, New York-London, 1968. MR0226634

[14] C. B. Thomas, *Characteristic classes and the cohomology of finite groups*, Cambridge Studies in Advanced Mathematics, vol. 9, Cambridge University Press, Cambridge, 1986. MR878978

[15] Charles A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994. MR1269324