Minimal models for monomial algebras

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Models of algebras

- A model of a dga algebra A is an epimorphism of dga algebras $B \longrightarrow A$ inducing an isomorphism on homology, where B is free as a graded algebra.
- The minimal model of an algebra is the “most economic” way to resolve it in Alg with respect to the size of B.
- The differential of a minimal model is, in general, *very complicated* to compute explicitly, although we have a general recipe to define it.

Why models?

- Models allow us to compute Yoneda (co)algebras, Hochschild (co)homology and cyclic homology, among other invariants.
- In the commutative setting, models have been successfully used to solve open problems, like Han's conjecture [2] and the rational Ganea conjecture [3].
- Obtaining models of other type of monoids, like operads, allows to describe its "homotopy algebras" in terms of the operations of the model.

The problem: resolve an algebra

Given a monomial algebra A, give

1. a graded vector space V,
2. a map $f : TV \to A$ and
3. a differential ∂ on TV so that

the induced map $Hf : H(TV) \to A$ is an *isomorphism*.

...so we are resolving A not as a bimodule, but as an *algebra*. The real problem is giving ∂ explicitly.

The plan

We will do this in a few steps.

1. Show that V is (isomorphic to) $\mathrm{Tor}_A := \mathrm{Tor}_A(\Bbbk, \Bbbk)$ if the model is minimal, and find a convenient basis for it.
2. Produce a datum that remembers how to compute Tor_A from the bar complex BA by killing cells.
3. Show that this datum determines a minimal model $(\Omega_\infty(\mathrm{Tor}_A), \partial)$ of A.
4. Give explicit formulas for ∂ on generators (Leibniz's rule does the rest).

Step 1. Resolving the ambiguities (the generators)

Indecomposables. The space generators of the minimal model of the algebra A is given by Tor_A. We can describe it in terms of the overlappings of relations of A, which we call *chains*:

The 0-chains C_0 are the variables of A, the 1-chains C_1 are the minimal relations of A, and for $r > 1$, the r-chains C_r are overlaps of r relations satisfying a recursive "minimality" condition.

Theorem (Anick and Green–Happel–Zacharia)

Chains give a basis of Tor_A for A monomial: for each $r \in \mathbb{N}_0$, we have that $\mathrm{Tor}_A^{r+1} \simeq \Bbbk C_r$.

Two examples

An N**-Koszul algebra**. The algebra $A = \Bbbk[t]/(t^N)$ has one r-chain for each $r \in \mathbb{N}_0$: t^{rN} is the unique $2r$-chain, and t^{rN+1} is the unique $2r+1$-chain.

A non 3**-Koszul algebra.** The quiver algebra

$$R = \{xy^2, y^2z\}$$

has finitely many chains: we have that $C_0 = \{x, y, z\}$, $C_1 = \{xy^2, y^2z\}$, $C_2 = \{xy^2z, xy^3z\}$ and that $C_r = \varnothing$ for $r > 2$.

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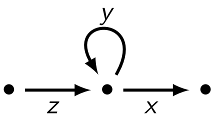

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Step 2. Algebraic discrete Morse theory (the retraction)

One can build a model of A easily by delooping: there is always a dga coalgebra BA and a quasi-isomorphism $\Omega BA \to A$. So we will instead find a minimal model of ΩBA.

The bar construction BA is a coalgebra, and we know how to obtain a model of ΩBA using $H(BA)$ as its generators. This requires finding a homotopy retract from BA to $H(BA)$.

Algebraic discrete Morse theory gives us this datum. It "remembers" how we went from BA to $H(BA) = \mathrm{Tor}_A$, its homology. Geometrically, it tells us how to "kill cells" in ΩBA to obtain a *minimal* model.

Anick from ADMT and the takeaway

Theorem (Jöllenbeck–Welker)

If B is a complex of free modules and M is a Morse matching, there is a complex B^M and maps

$$i : B^M \to B, \quad p : B \to B^M$$

such that $pi = 1_{B^M}$ and ip is homotopic to the identity of B.

In [4], the authors give a Morse matching M on BA so that $(BA)^M$ has basis the set of Anick chains, which we know how to identify with Tor_A. This gives us maps $i : \mathrm{Tor}_A \to BA$, $p : BA \to \mathrm{Tor}_A$ and $h : BA \to BA$ as above.

The takeaway from Step 2.

Relevant data. For our purposes, it suffices to understand here

- The homotopy $h : BA \longrightarrow BA$ afforded by ADMT.
- Its behaviour with respect to $\Delta : BA \longrightarrow BA \otimes BA$.
- How these two operators act on Tor_A.

Indeed, the following result describes the differential in the minimal model with generators Tor_A in terms of these two operators.

Step 3. Homotopy transfer theorem (the higher structure)

Notation: We write $\Omega_\infty(\mathrm{Tor}_A)$ for the free algebra over $s^{-1}\overline{\mathrm{Tor}_A}$.

Homotopy Transfer Theorem

Let B be a dga coalgebra (ex. $B = BA$) and consider a contraction as before. There exists a differential ∂ on $\Omega_\infty(B^M)$ and a homotopy retract data of dga algebras from ΩB to $\Omega_\infty(B^M)$.

This gives us what we want: a map $\Omega_\infty(\mathrm{Tor}_A) \longrightarrow \Omega BA$ which is, in fact, a homotopy equivalence. One easily checks $(\Omega_\infty(\mathrm{Tor}_A), \partial)$ is minimal. But how to compute ∂?

The differential in terms of trees

For a given planar binary tree T, let Δ_T be the operator obtained by decorating the root of T by i (inclusion $\mathrm{Tor}_A \subseteq BA$), its leaves by p (projection $BA \to Tor_A$), its internal edges by h, and its vertices by Δ. These define maps from Tor_A to $\Omega_\infty(\mathrm{Tor}_A)$.

Theorem (Markl [5] and many others before...)

We have that $\partial|_{\mathrm{Tor}_A} = \sum_{n\geqslant 2} \Delta_n$ where for each $n \in \mathbb{N}_0$, $\Delta_n = \sum_T (-1)^{\vartheta(T)} \Delta_T$ as T ranges through all planar binary trees with n leaves.

Step 4. The exchange rule (reduction to one tree)

The explicit description of the differential in $\Omega_\infty(\mathrm{Tor}_A)$ was simplified with the following vanishing and exchange rules:

Proposition (T.)

If h is the homotopy $h : BA \longrightarrow BA$ and Δ is the (deconcatenation) of BA, then

$$\Delta h = (h \otimes 1)\Delta \mod (\mathrm{Tor}_A \otimes BA).$$

Moreover, h vanishes on Tor_A, and $h^2 = 0$. All this implies that $\Delta_T = 0$ unless T is a *right comb*.

Step 4. The action of the comb

Decompositions. If γ is an r-chain in Tor_A, a *decomposition* of γ is a sequence $(\gamma_1, \ldots, \gamma_n)$ of chains of length $r_1, \ldots, r_n$ such that

- $r_1 + \cdots + r_n = r - 1$ (correct degree),
- $\gamma = \gamma_1 \ldots \gamma_n$ (as monomials).

We say the decomposition has length n, and call it an n-decomposition.

Example. The 2-chain xy^2z has two 2-decompositions, (x, y^2z) and (xy^2, z), and no other decompositions. The 1-chain xy^2 has a single 3-decomposition, (x, y, y), and no other decompositions.

Theorem (T.)

If T is the right comb and γ is any chain, then $\Delta_T(\gamma)$ is a signed sum through all decompositions of γ.

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Description of the model

The end result is the following description of minimal models of monomial quiver algebras.

Theorem (T.)

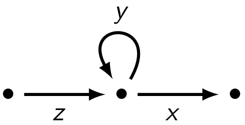
For each monomial algebra A there is a minimal model $\mathcal{M} \longrightarrow A$ where $\mathcal{M} = (\Omega_\infty \mathrm{Tor}_A(\mathbb{k},\mathbb{k}), \partial)$ is the ∞-cobar construction on $\mathrm{Tor}_A(\mathbb{k},\mathbb{k})$. For a chain γ,

$$\partial(\gamma) = \sum_{n\geqslant 2} (-1)^{\binom{n+1}{2}+|\gamma_1|-1} \gamma_1 \cdots \gamma_n,$$

where the sum ranges through all possible decompositions of γ.

Some examples

The quiver with monomial relations



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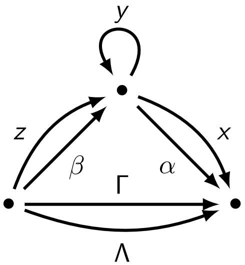
has minimal model given by the following dg quiver



$$|x| = |y| = |z| = 0$$

$$|\alpha| = |\beta| = 1$$

$$|\Lambda| = |\Gamma| = 2$$

$$\partial x = \partial y = \partial z = 0$$

$$\partial\alpha = xy^2, \partial\beta = y^2z$$

$$\partial\Gamma = \alpha z - x\beta$$

$$\partial\Lambda = xy\beta - \alpha yz$$

A non-monomial example

The *super Jordan plane* is the associative algebra

$$\Bbbk\langle x, y | x^2, x^2y - yx^2 - xyx\rangle.$$

Its minimal models has two generators x_n, y_n in degree n for each $n \in \mathbb{N}_0$ corresponding to the ambiguities x^{n+1} and y^2x^n. Its differential vanishes on degree 0 and for $n \in \mathbb{N}_0$ is as follows:

$$\partial y_{n+1} = y^2x_n - x_ny^2 - \sum_{s+t=n} x_s y x_t - \sum_{\substack{s+t=n \\ t\geqslant 1}} (x_s y_t - (-1)^t y_t x_s),$$

$$\partial x_{n+1} = \sum_{s+t=n} (-1)^s x_s x_t.$$

Application: the Gerstenhaber bracket

If $B \to A$ is a model of an algebra A, one can compute Hochschild cohomology of A through derivations $B \longrightarrow B$.

Fact. The Gerstenhaber bracket in Hochschild cohomology is just the Lie bracket on derivations. So one can compute it easily if one can compute $\mathrm{HH}^*(A)$ through a model.

One can also compute Hochschild homology and cyclic homology, and other gadgets associated to these invariants.

Thank you!

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