Solutions to Assignment 6

Normal subgroups, quotients and the isomorphism theorems

MAU22101 — Group Theory

NAME AND SURNAME: ....

STUDENT NUMBER: .... NUMBER OF PAGES: ....

Note. Solutions to this assignment are **due** by 3:00 pm on Thursday, November 7th. Remember to **fill in** all the information above and **staple** all your sheets together, including this one. All exercises are weighed equally unless otherwise stated.

Recollections. We fix a group G throughout this assignment. Recall that an *automorphism* is an isomorphism of a group G with itself. Denote by $\mathsf{Aut}(G)$ the set of automorphisms of G. Composition of maps gives a group operation on $\mathsf{Aut}(G)$. The *centre* of G is the subgroup $\mathsf{Z}(G) = \{g \in G \mid gx = xg\ \forall x \in G\}$.

Exercise 1. For each $g \in G$, let $\mathrm{ad}_g : G \longrightarrow G$ be the map such that $\mathrm{ad}_g(h) = ghg^{-1}$.

1. Show that for each $g \in G$, this defines an automorphism of G. We call it an *inner automorphism of G*.
2. Show that the map $\mathrm{ad} : G \longrightarrow \mathsf{Aut}(G)$ is a group homomorphism. We call its image the *group of inner automorphisms of G* and denote it $\mathsf{Inn}(G)$.
3. Show that $\mathsf{Inn}(G)$ is a normal subgroup of $\mathsf{Aut}(G)$. The quotient group that results is called the *group of outer automorphisms of G* and we write it $\mathsf{Out}(G)$.

Bonus: Show that the center $\mathsf{Z}(G)$ is normal in G and that the group of inner automorphisms of G is isomorphic to the quotient group $G/\mathsf{Z}(G)$.

Solution 1. Observe that the inverse map to ad_g is ad_h where $h = g^{-1}$. One checks that $\mathrm{ad}_g(ab) = \mathrm{ad}_g(a)\mathrm{ad}_g(b)$ and that $\mathrm{ad}_g(1) = 1$, provin that this is an automorphism of G. Observe now that $\mathrm{ad}_{gh} = \mathrm{ad}_g \circ \mathrm{ad}_h$, meaning that ad is a group homomorphism. Its kernel consists of those $g \in G$ for which $\mathrm{ad}_g(h) = h$ for all h, and this means precisely that $g \in \mathsf{Z}(G)$. Its image consists of those inner automorphisms of G, so the first isomorphism theorem gives the bonus result that $G/\mathsf{Z}(G) \simeq \mathsf{Inn}(G)$. Finally, inner automorphisms form a normal subgroup, since for any automorphism φ we have that $\varphi \circ \mathrm{ad}_g \circ \varphi^{-1} = \mathrm{ad}_{\varphi(g)}$.

Exercise 2. Let N and M be normal subgroups of G and assume that $NM = G$ and $N \cap M = 1$. Show the map $f : N \times M \longrightarrow G$ such that $f(n,m) = nm$ is an isomorphism of groups. **Hint:** to show that any $n \in N$ and $m \in M$ commute prove that $nmn^{-1}m^{-1} \in N \cap M$ and use this to show f is a group homomorphism.

Solution 2. Since $G = NM$, f is surjective. Moreover, if $n \in N$ and $m \in M$, then $nmn^{-1}m^{-1} \in M \cap N$, since it is both of the form $(nmn^{-1})m^{-1}$ and $n(mn^{-1}m^{-1})$. Thus elements of M and N commute. This means that f is a group homomorphism, and that its kernel consists of those (m,n) such that $mn = 1$. This means, naturally, that $m = n^{-1}$ is an element of $N \cap M$. Thus, f has trivial kernel so f is an isomorphism, which is what we wanted.

Exercise 3.

1. Suppose that G is finite. Show that a subgroup H of G is normal if and only if it is stable under conjugation by any element of G.
2. Let $\langle M, N\rangle = K \leqslant \mathsf{GL}(2, \mathbb{Q})$ for $M = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $N = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, Show that $H = \langle M\rangle$ is stable under conjugation by any element of K, but that it is not normal in K.

Solution 3. Pick $g \in G$ and assume that $gHg^{-1} \subseteq H$. Since H is finite and since conjugation is a bijection, equal cardinality of H and gHg^{-1} forces equality, hence H is normal. On the other hand, if we conjugate M by N we get the matrix M^2. This generates the proper subgroup $2\mathbb{Z}$ of $H \simeq \mathbb{Z}$, so H is not normal in K even though it is stable by conjugation.

Exercise 4.

1. Show that if H is a normal subgroup of G of index $n \in \mathbb{N}$, then for each $g \in G$ we have that $g^n \in H$. **Hint:** consider the class of g in the finite group G/H.
2. Show that if H is a subgroup of index two in G, then H is normal in G. **Hint:** if $g \notin G$, then $\{gH, H\}$ and $\{H, Hg\}$ both form a partition of G.

Solution 4. In the quotient group G/H of order n, every element x is such that $x^n = e$ by Lagrange's theorem. Hence taking the $x = [g]$ the class of g, we get $g^n \in H$, as we wanted. For the second claim, we observe as in the **Hint** that both $\{H, gH\}$ and $\{H, Hg\}$ form a partition of G when $g \notin H$. But this means, of course, that $gH = Hg$, so H is normal.