Solutions to Assignment 5

Cosets, Lagrange's theorem and cyclic groups

MAU22101 — Group Theory

NAME AND SURNAME: ....

STUDENT NUMBER: .... NUMBER OF PAGES: ....

Note. Solutions to this assignment are **due** by 3:00 pm on Thursday, October 31th. Remember to **fill in** all the information above and **staple** all your sheets together, including this one. All exercises are weighed equally unless otherwise stated.

Recollections. Recall that the cartesian product of the sets X and Y is the set $X \times Y$ of ordered pairs (x, y), where $x \in X$ and $y \in Y$.

Exercise 1. Let G be a group and suppose that H is a subgroup of index 2.

1. Show that if $g \in G$ has odd order, then $g \in H$.
2. Let A_4 be the subgroup of even permutation of S_4. Without using **Exercise 3** show that it contains no subgroup of order 6. Thus, the converse of Lagrange's theorem does not hold.

Hint: if $g \notin H$, then H and gH are all cosets of H in G. Consider the cases $g^2 \in H$ or $g^2 \in gH$ and obtain a contradiction.

Solution 1. For the first part, let us use the hint. If g has odd order and $g \notin H$, then either $g^2 \in H$ or $g^2 \notin H$. In the first case, since some even power of g is its inverse, we get that $g^{-1} \in H$, so $g \in H$. This contradicts out assumption that $g \notin H$. Similarly, if $g^2 \in gH$, then $g^2 = gh$ for some $h \in H$, so that $g = h \in H$, which is, again, a contradiction. We conclude that $g \in H$ whenever $g \in G$ has odd order. For the second part, we check that A_4 contains 8 elements of order three and has order 12. Since a subgroup of order 6 then has index 2 in A_4, it would contain at least 9 elements, which is of course impossible. Other ways of solving this exercise can be found in these short notes of Keith Conrad.

Exercise 2. Let G and K be groups.

1. Prove that $G \times K$ is a group for product $(g, k)(g', k') = (gg', kk')$. We call $G \times K$ the *direct product of G and K*.
2. Show that for every integer n the group $\mathbb{Z}/n \times \mathbb{Z}/n$ is not cyclic.

Note: when $n = 2$, the group we obtain is isomorphic to the Klein group introduced in Assignment 4.

Solution 2. All group axioms for $G \times K$ follow by applying the group axioms to G and K coordinate-wise. That is, the product is associative because it is associative in each coordinate. The unit is $(1, 1)$ and the inverse of (g, h) is (g^{-1}, h^{-1}). For the second part, we observe every element in this group has order a divisor of n, since for any (x, y) in the product group we have $(x, y)^n = (x^n, y^n) = (1, 1)$. Since the group has order n^2 but contains no element of order n^2, it cannot be cyclic.

Exercise 3. Determine all 8 non-trivial subgroups of A_4, the alternating group on four letters. **Note:** write them out explicitly. You should obtain three cyclic subgroups of order two, one Klein subgroup and four cylic subgroups of order three.

Solution 3. The group A_4 has order 12, so by Lagrange's theorem it can only have subgroups of order $1,2,4,6$ or 12. By **Exercise 1** we know there are no subgroups of order 6. There are three elements in A_4 of order 2, namely $(12)(34)$, $(13)(24)$ and $(14)(23)$. These account for all three subgroups of order 2. These three double transpositions generate, in turn, a subgroup of order four isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$, and this is the only subgroup of order four of A_4. Indeed, there are no elements of order four in A_4 since these can only be four cycles, and these are odd. So we are done looking for subgroups of order four. Finally, there are four 3-cycles in A_4, namely, $(123), (124), (234)$ and (134), and each of these generates a cyclic subgroup of order three. This gives all non-trivial subgroups of A_4.

Exercise 4. Let H_1 and H_2 be subgroups of a finite group G and suppose that their orders are coprime. Show that $H_1 \cap H_2$ is the trivial group.

Solution 4. By Lagrange's theorem, the group $H_1 \cap H_2$ has order d that divides both that of H_1 and that of H_2. It follows that d divides the greatest common divisor of their orders. Since these are coprime, it follows that $d = 1$, so that $H_1 \cap H_2$ is the trivial group.