# Solutions to Assignment 4

Homomorphisms and subgroups

MAU22101 — Group Theory

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STUDENT NUMBER: .................................. NUMBER OF PAGES: ..................................

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**Note.** Solutions to this assignment are **due** by 3:00 pm on Thursday, October 17th. Remember to **fill in** all the information above and **staple** all your sheets together, including this one. All exercises are weighed equally unless otherwise stated.

**Recollections.** Recall that a subset $H$ of a group $G$ is a *subgroup* if it contains the identity element 1 of $G$ and it is closed under multiplication and inverses. A group $G$ is called abelian if its product is commutative, otherwise, it is called non-abelian. A group is cyclic if it is generated by one element. If $S$ is a subset of $G$, we write $\langle S\rangle$ for the subgroup generated by the elements of $S$.

**Exercise 1.** The Klein four group $K$ is the four element group generated by the transpositions $t = (12)$ and $s = (34)$ in the symmetric group $S_4$.

1. Show that $t^2 = s^2 = 1$ and that $st = ts$, and explain why this shows that $K = \{1, s, t, ts\}$.
2. Write down the multiplication table for $K$.
3. Show show that $K$ is not cyclic, and conclude that $K$ is not isomorphic to $\mathbb{Z}/4$.

**Solution 1.** The first two items are a direct computation. To see that $K$ has four elements, we note that the rule $st = ts$ allows us to write any word in $s, t$ in the form $t^N s^M$. Since $s, t$ have order two, we can assume $N, M \in \{0, 1\}$. This gives the four elements we were after. We omit writing down the multiplication table of $K$. For the last item, $K$ is of order 4, so were it cyclic there should exist an element in $K$ of order 4. But we already noted all elements in $K$ are of order 2 or less.

**Exercise 2.** Show that a subset $H$ of a group $G$ is a subgroup if, and only if, $1 \in H$ and $x, y \in H$ implies that $xy^{-1} \in H$.

**Solution 2.** If $H$ is a subgroup then $1 \in H$, and if $x, y \in H$ then $x, y^{-1} \in H$, so that $xy^{-1} \in H$. Conversely, suppose that $1 \in H$ and the condition holds. Taking $x = 1$ we see that $H$ is closed under inverses. Then for $x, y \in H$ we get $x, y^{-1} \in H$, so the condition gives that $xy \in H$, and $H$ is closed under products.

**Exercise 3.** Let $G$ be a group, and let $x \in G$. Show that $x$ has order $n$ if only if, $\langle x\rangle$ has $n$ elements. **Note:** the implication ($\Rightarrow$) was shown in class, so you only have to show the second condition implies the first.

**Solution 3.** Suppose that $\langle x \rangle$ has $n$ elements. By the pigeonhole principle, among the $n+1$ elements $1, x, \ldots, x^{n+1}$ there must exist two that are equal. Then $x^{j-i} = 1$ for some $i < j$. This means $x$ has some finite order $k$. But by the result in class, we know that then $\langle x \rangle$ has order $k = n$.

**Exercise 4.** Consider the $2 \times 2$ invertible matrices over the complex numbers given by

$$A = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

1. Show that the subgroup generated by them has 8 elements.
2. **Bonus:** show that this group is non-abelian and contains a unique subgroup of order two.

**Hint:** compute powers of $A$, powers of $B$, and $AB$ and $BA$. Be careful about repetitions!

**Solution 4.** We compute that $A^2 = B^2 = -1$, so $A^3 = -A$ and $B^3 = -B$, and $A$ and $B$ have order 4. We also compute that $AB = -BA$, so that the same argument as in **Exercise 1** shows that we can write a generic element of $\langle A, B \rangle$ as $\pm A^p B^q$ with $p, q \in \{0, 1\}$. This accounts for at most eight elements, and by inspection we have exactly eight. The group is non-abelian since $AB = -BA$, and the subgroup $\{1, -1\}$ is the unique subgroup of order 2 since any other element not in this set has order 4. If we set $A = i$, $B = j$ and $AB = k$, we see that we have constructed Hamilton's quaternion group, which we will write $Q_8$.