Solutions to Assignment 2

Divisibility and induction

MAU22101 — Group Theory

NAME AND SURNAME: ....
STUDENT NUMBER: .... NUMBER OF PAGES: ....

Note. Solutions to this assignment are **due** by 3:00 pm on Thursday, October 3rd. Remember to **fill in** all the information above and **staple** all your sheets together, including this one. All exercises are weighed equally unless otherwise stated.

Recollections. Recall that for every $n, k \in \mathbb{N}$ the *binomial number* $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, and that the greatest common divisor of a tuple (a, b) is the unique positive integer d with the property that $d \mid a$, $d \mid b$ and if $d' \mid a$ and $d' \mid b$ then $d' \mid d$.

Exercise 1. Show that for each natural number n, we have that 2 divides $\binom{2n}{n}$, and that $n+1$ also divides $\binom{2n}{n}$.

Solution 1. One can prove by direct algebraic manipulation that for each $n \in \mathbb{N}$ we have that

$$\frac{1}{2}\binom{2n}{n} = \binom{2n-1}{n}, \quad \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1}$$

and the right hand side of each equality is manifestly an integer, which is all that we need.

Exercise 2. Show that for any integer a the greatest common divisor between $5a+8$ and $7a+3$ is either 1 or 41. Find an integer so that it equals 1, and check that for $a = 23$, this number is indeed 41.

Solution 2. We can transform the pair $(5a+8, 7a+3)$ into the pair $(a+18, -41)$ by successive subtractions from one of the coordinates onto the other. If $a+18$ is coprime to 41, which is a prime, the result is 1, and else it is 41. We can choose $a = 0$ to get 1. For $a = 23$, we get $a+18 = 41$, so the common divisor is 41.

Exercise 3. Following the extended Euclidean algorithm, find a pair of integers (x, y) such that $123x + 164y = 41$. Record all steps of the extended algorithm in the form of a table, or otherwise.

Solution 3. This is actually very simple, since $164 - 123 = 41$. We will be more careful next time not to hand out non-exercises!

Exercise 4.

1. Show by induction that for every $n \in \mathbb{N}$ we have that $1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$.
2. Show by induction that for every $n \in \mathbb{N}$ we have that $1^3 + 2^3 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2$.
3. **Bonus:** show by induction that for every natural number n the sum $1^p + \cdots + n^p$ is a polynomial in n of degree $p+1$.

Solution 4. In what follows, we will denote by $f(n)$ the sum to the left and by $g(n)$ the formula to the right, so we seek to show that these two are equal.

1. We check that $f(1) = g(1) = 1$, and that $g(n+1) - g(n) = (n+1)^2$. By induction, it follows that $g(n) = f(n)$ for all n, since $f(n+1) - f(n) = (n+1)^2$, too.
2. As in the previous item, we check that $f(1) = g(1) = 1$, and that $g(n+1) - g(n) = (n+1)^3$. By induction, it follows that $g(n) = f(n)$ for all n, since $f(n+1) - f(n) = (n+1)^3$, too.
3. We will show that the sum $S(n)$ is a polynomial in n of degree $p+1$ and leading coefficient $\frac{1}{n+1}$, that is,

$$S(n) := 1^p + \cdots n^p = \frac{1}{n+1} n^{p+1} + h(n)$$

where h is a polynomial of degree p or less. To do this, we assume by induction on p that our claim is true (observing it is true for $p = 1$), where $S_1(n) = \frac{1}{2}n^2 + \frac{1}{2}n$. Now write

$$(j+1)^{p+1} - j^{p+1} = \sum_{i=0}^{p} \binom{p+1}{i} j^i$$

And let us take the sum of this through all $j \in \{1, \dots, n\}$. We obtain that

$$(n+1)^{p+1} - 1 = \sum_{i=0}^{p} \binom{p+1}{i} S_i(n).$$

By hypothesis, all the terms to the right except possibly $S_p(n)$ are polynomials of the appropriate degrees. By solving for $S_p(n)$ we obtain the result, and being careful about the coefficients, we obtain the leading coefficient of $S_p(n)$ is $\frac{1}{n+1}$. *Note:* If this is a bit hard to follow, carry out the computations for a few values of p to obtain, without guessing, the formulas for items 1. and 2.: this should help observe what is going on.