Solutions to Assignment 1

Preliminaries

MAU22101 — Group Theory

NAME AND SURNAME: ....

STUDENT NUMBER: .... NUMBER OF PAGES: ....

Note. Solutions to this assignment are **due** by 3:00 pm on Thursday, September 26th. Remember to **fill in** all the information above and **staple** all your sheets together, including this one. All exercises are weighed equally unless otherwise stated.

Recollections. Recall that the *support* of a permutation σ is the set $X(\sigma) := \{i : \sigma(i) \neq i\}$. For example, $X(\mathrm{id}) = \varnothing$, while for $X((12)(34)) = \{1,2,3,4\}$. If σ is a permutation, a *power of* σ is any permutation of the form σ^k for some $k \in \mathbb{N}$. Recall that the *sign* of a permutation σ can be computed as the determinant of the matrix $M(\sigma)$ with $M(\sigma)_{ij} = 0$ unless $j = \sigma(i)$, in which case $M(\sigma)_{ij} = 1$. Alternatively, the sign of σ is 1 if it can be written as a product of an even number of transpositions, and -1 if it can be written as a product of an odd number of transpositions. For example, $\mathrm{sign}(123) = 1$ while $\mathrm{sign}(12) = -1$.

Exercise 1. By considering the cycle (1234) in S_4, show that the power of a cycle may not be a cycle. Compute all three distinct powers of (132) for a non-counterexample.

Solution 1. We compute that $(1234)^2 = (13)(24)$, which is not a cycle. We have that $(132)^2 = (123)$ which is a cycle, and $(132)^3 = 1$ which is also a cycle (an empty one).

Exercise 2. Two permutations σ and τ of S_n are called *disjoint* if the sets $X(\sigma)$ and $X(\tau)$ are disjoint. Show that in this case, τ and σ commute, that is, $\tau\sigma = \sigma\tau$.

Solution 2. Note that if i is fixed by both τ and σ we have that $\tau\sigma(i) = \sigma\tau(i) = i$, so we can assume that i is moved by τ, for example —the proof is the same if i is moved by σ, by symmetry. Since $i \in X(\tau)$, then $\sigma(i) = i$ and, moreover, $\tau(i)$ is fixed by σ. Indeed, $X(\tau)$ is preserved by τ, for if $\tau\tau(i) = \tau(i)$ then cancelling τ gives $\tau(i) = i$, and this cannot be. We conclude that $\tau(i) \in X(\tau)$ which is disjoint from $X(\sigma)$, and then that $\sigma\tau(i) = \tau(i) = \tau\sigma(i)$, which is what we wanted.

Exercise 3. Draw a table with the sign of each of the six permutations of S_3. *Hint:* show that the sign of a cycle of even length is odd, and the sign of a cycle of odd length is even.

Solution 3. The three permutations 1,(123) and (132) have sign 1, while the three permutations (12),(23) and (13) have sign -1. To prove the hint, which then gives the desired answer, for every element of S_3 is a cycle, we note that if $\sigma = (i_1 \dots i_n)$ is a cycle can write it is a product of $n-1$ transpositions as follows: $\sigma = (i_{n-1} i_n \cdots (i_2 i_n)(i_1 i_n)$, so its sign is $(-1)^{n-1}$. It is clear then that an odd permutation has even sign and that an even permutation has odd sign, which is what we wanted. One can, of course, compute the signs by hand using 3×3 matrices, or exhibit decompositions of the only two 3-cycles in S_3 as products of two transpositions.

Exercise 4. Let $f : X \to Y$ be a function and let $A, A' \subseteq X$ and $B, B' \subseteq Y$. Prove the following:

1. $f\left(f^{-1}(B)\right) \subseteq B$ and $A \subseteq f^{-1}\left(f(A)\right)$.
2. If $B \subseteq B'$, then $f^{-1}(B) \subseteq f^{-1}(B')$, and if $A \subseteq A'$, then $f(A) \subset f(A')$.
3. $f^{-1}\left(B \cup B'\right) = f^{-1}(B) \cup f^{-1}(B')$ and $f^{-1}\left(B \cap B'\right) = f^{-1}(B) \cap f^{-1}(B')$.
4. $f(A \cup A') = f(A) \cup f(A')$ and $f\left(A \cap A'\right) \subseteq f(A) \cap f(A')$, but give a counterexample to show the converse inclusion fails.

Solution 4.

1. Elements of $f^{-1}(B)$ have, by definition, image in B. This is what the inclusion $f(f^{-1}(B)) \subseteq B$ is stating. Dually, every element of A maps under f to $f(A)$. This is what the inclusion $A \subseteq f^{-1}(f(A))$ is stating.
2. Every element of X that maps into B maps into B', since $B \subseteq B'$. Hence $f^{-1}(B) \subseteq f^{-1}(B')$. Since every element of A is contained in A', every element of the form $f(a)$ for $a \in A$ is also of the form $f(a')$ for $a' \in A'$, and this means $f(A) \subseteq f(A')$.
3. If an element of X maps to B or to B' then, by definition, it is an element of X that maps to B or an element of X that maps to B'. This is precisely what the equality $f^{-1}(B \cup B') = f^{-1}(B) \cup f^{-1}(B')$ means. Replacing "or" with "and" gives us the second equality.
4. An element of Y of the form $f(x)$ for $x \in A$ or $x \in A'$, is the same as an element of Y that is for the form $f(x)$ for $x \in A$ or an element of Y that is of the form $f(x)$ for $x \in A'$. This gives us the first equality. On the other hand, an element of Y of the form $f(x)$ for $x \in A$ and $x \in A'$ is, in particular, an element of Y of the form $f(x)$ for $x \in A$ and an element of Y of the form $f(x)$ for $x \in A'$. However, if we consider the only function $f : \{0,1\} \longrightarrow \{0\}$ and the subsets $\{0\} = A'$ and $\{1\} = A$ of the domain, we see that their intersection is empty, even though $f(A') = f(A) = \{1\}$ is nonempty. We conclude the inclusion can be strict.