Assignment 7

Group actions

MAU22101 — Group Theory

NAME AND SURNAME: ....

STUDENT NUMBER: .... NUMBER OF PAGES: ....

Note. Solutions to this assignment are **due** by 3:00 pm on Thursday, November 28th. Remember to **fill in** all the information above and **staple** all your sheets together, including this one. All exercises are weighed equally unless otherwise stated. Attempt **two** out of the three exercises, and **cross out** from this cover sheet the one you will **not attempt.**

Recollections. We fix a finite group G and X a set with a G-action. The *orbit* of an element $x \in X$ is the set $O_x = \{gx : g \in G\}$. These form a partition of X, and we write X/G for the set $\{O_x : x \in X\}$ of G-orbits of X. The *stabilizer* of an element $x \in X$ is the set $G_x = \{g \in G : gx = x\}$.

Exercise 1. Recall we defined the quaternion group Q_8 in **Assignment 4.**

1. Determine all interior automorphisms of Q_8. They are isomorphic to K_4.
2. Exhibit an automorphism of Q_8 that is *not* inner.
3. Show that an automorphism of Q_8 fixes -1 and is determined by the image of i and j. Hence show that there are at most 24 automorphisms of Q_8.
4. **Bonus:** show that the group of automorphisms of Q_8 is isomorphic to S_4.

Exercise 2. Consider the action of G on itself by conjugation. Show that, for each $x \in G$:

1. The stabilizer G_x is the collection of $g \in G$ that commute with x. We call this subgroup the *centralizer of x* and write it C_x.
2. For each $x \in G$, the orbit O_x has exactly one element if and only if $x \in \mathsf{Z}(G)$.
3. Conclude that $|G| = |\mathsf{Z}(G)| + \sum_x [G : C(x)]$ where the sum runs through a set of representatives for non-central elements. We call this the *class equation* of G.
4. **Bonus:** show that if G has p^n elements for some n, then it has non-trivial center.

Exercise 3. For each $g \in G$, let $\mathsf{Fix}(g)$ be the collection of elements $x \in X$ such that $gx = x$, and let $\mathsf{fix}(g)$ denote its cardinality. Consider the set $S = \{(g, x) \in G \times X : gx = x\}$.

1. Show that for each $g \in G$ there are $\mathsf{fix}(g)$ many pairs in S of the form (g, x).
2. Show that for each $x \in X$, there are $|G_x|$ many pairs in S of the form (g, x).
3. Conclude that $\sum_{g \in G} \mathsf{fix}(g) = \sum_{x \in X} |G_x|$.
4. **Bonus:** use the Orbit–Stabilizer theorem to show that

$$\frac{1}{|G|} \sum_{g \in G} \mathsf{fix}(g) = |X/G| \qquad ,$$

   That is, the average number of fixed points of G equals the number of orbits of the action.