Assignment 6

Normal subgroups, quotients and the isomorphism theorems

MAU22101 — Group Theory

NAME AND SURNAME: ....

STUDENT NUMBER: .... NUMBER OF PAGES: ....

Note. Solutions to this assignment are **due** by 3:00 pm on Thursday, November 7th. Remember to **fill in** all the information above and **staple** all your sheets together, including this one. All exercises are weighed equally unless otherwise stated.

Recollections. We fix a group G throughout this assignment. Recall that an *automorphism* is an isomorphism of a group G with itself. Denote by $\mathsf{Aut}(G)$ the set of automorphisms of G. Composition of maps gives a group operation on $\mathsf{Aut}(G)$. The *centre* of G is the subgroup $\mathsf{Z}(G) = \{g \in G \mid gx = xg \ \forall x \in G\}$.

Exercise 1. For each $g \in G$, let $\mathrm{ad}_g : G \longrightarrow G$ be the map such that $\mathrm{ad}_g(h) = ghg^{-1}$.

1. Show that for each $g \in G$, this defines an automorphism of G. We call it an *inner automorphism of G*.
2. Show that the map $\mathrm{ad} : G \longrightarrow \mathsf{Aut}(G)$ is a group homomorphism. We call its image the *group of inner automorphisms of G* and denote it $\mathsf{Inn}(G)$.
3. Show that $\mathsf{Inn}(G)$ is a normal subgroup of $\mathsf{Aut}(G)$. The quotient group that results is called the *group of outer automorphisms of G* and we write it $\mathsf{Out}(G)$.

Bonus: Show that the center $\mathsf{Z}(G)$ is normal in G and that the group of inner automorphisms of G is isomorphic to the quotient group $G/\mathsf{Z}(G)$.

Exercise 2. Let N and M be normal subgroups of G and assume that $NM = G$ and $N \cap M = 1$. Show the map $f : N \times M \longrightarrow G$ such that $f(n, m) = nm$ is an isomorphism of groups. **Hint:** to show that any $n \in N$ and $m \in M$ commute prove that $nmn^{-1}m^{-1} \in N \cap M$ and use this to show f is a group homomorphism.

Exercise 3.

1. Suppose that G is finite. Show that a subgroup H of G is normal if and only if it is stable under conjugation by any element of G.
2. Let $\langle M, N\rangle = K \leqslant \mathsf{GL}(2, \mathbb{Q})$ for $M = \left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$ and $N = \left(\begin{smallmatrix}2 & 0\\ 0 & 1\end{smallmatrix}\right)$, Show that $H = \langle M\rangle$ is stable under conjugation by any element of K, but that it is not normal in K.

Exercise 4.

1. Show that if H is a normal subgroup of G of index $n \in \mathbb{N}$, then for each $g \in G$ we have that $g^n \in H$. **Hint:** consider the class of g in the finite group G/H.
2. Show that if H is a subgroup of index two in G, then H is normal in G. **Hint:** if $g \notin G$, then $\{gH, H\}$ and $\{H, Hg\}$ both form a partition of G.