Assignment 5

Cosets, Lagrange's theorem and cyclic groups

MAU22101 — Group Theory

NAME AND SURNAME: ....

STUDENT NUMBER: .... NUMBER OF PAGES: ....

Note. Solutions to this assignment are **due** by 3:00 pm on Thursday, October 31th. Remember to **fill in** all the information above and **staple** all your sheets together, including this one. All exercises are weighed equally unless otherwise stated.

Recollections. Recall that the cartesian product of the sets X and Y is the set $X \times Y$ of ordered pairs (x, y), where $x \in X$ and $y \in Y$.

Exercise 1. Let G be a group and suppose that H is a subgroup of index 2.

1. Show that if $g \in G$ has odd order, then $g \in H$.
2. Let A_4 be the subgroup of even permutation of S_4. Without using **Exercise 3** show that it contains no subgroup of order 6. Thus, the converse of Lagrange's theorem does not hold.

Hint: if $g \notin H$, then H and gH are all cosets of H in G. Consider the cases $g^2 \in H$ or $g^2 \in gH$ and obtain a contradiction.

Exercise 2. Let G and K be groups.

1. Prove that $G \times K$ is a group for product $(g, k)(g', k') = (gg', kk')$. We call $G \times K$ the *direct product of G and K*.
2. Show that for every integer n the group $\mathbb{Z}/n \times \mathbb{Z}/n$ is not cyclic.

Note: when $n = 2$, the group we obtain is isomorphic to the Klein group introduced in Assignment 4.

Exercise 3. Determine all 8 non-trivial subgroups of A_4, the alternating group on four letters.
Note: write them out explicitly. You should obtain three cyclic subgroups of order two, one Klein subgroup and four cylic subgroups of order three.

Exercise 4. Let H_1 and H_2 be subgroups of a finite group G and suppose that their orders are coprime. Show that $H_1 \cap H_2$ is the trivial group.