Assignment 4

Homomorphisms and subgroups

MAU22101 — Group Theory

NAME AND SURNAME: ....

STUDENT NUMBER: .... NUMBER OF PAGES: ....

Note. Solutions to this assignment are **due** by 3:00 pm on Thursday, October 17th. Remember to **fill in** all the information above and **staple** all your sheets together, including this one. All exercises are weighed equally unless otherwise stated.

Recollections. Recall that a subset H of a group G is a *subgroup* if it contains the identity element 1 of G and it is closed under multiplication and inverses. A group G is called abelian if its product is commutative, otherwise, it is called non-abelian. A group is cyclic if it is generated by one element. If S is a subset of G, we write $\langle S \rangle$ for the subgroup generated by the elements of S.

Exercise 1. The Klein four group K is the four element group generated by the transpositions $t = (12)$ and $s = (34)$ in the symmetric group S_4.

1. Show that $t^2 = s^2 = 1$ and that $st = ts$, and explain why this shows that $K = \{1, s, t, ts\}$.
2. Write down the multiplication table for K.
3. Show show that K is not cyclic, and conclude that K is not isomorphic to $\mathbb{Z}/4$.

Exercise 2. Show that a subset H of a group G is a subgroup if, and only if, $1 \in H$ and $x, y \in H$ implies that $xy^{-1} \in H$.

Exercise 3. Let G be a group, and let $x \in G$. Show that x has order n if only if, $\langle x \rangle$ has n elements.
Note: the implication ($\Rightarrow$) was shown in class, so you only have to show the second condition implies the first.

Exercise 4. Consider the 2×2 invertible matrices over the complex numbers given by

$$A = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

1. Show that the subgroup generated by them has 8 elements.
2. **Bonus:** show that this group is non-abelian and contains a unique subgroup of order two.

Hint: compute powers of A, powers of B, and AB and BA. Be careful about repetitions!