Assignment 2

Divisibility and induction

MAU22101 — Group Theory

NAME AND SURNAME: ....

STUDENT NUMBER: .... NUMBER OF PAGES: ....

Note. Solutions to this assignment are **due** by 3:00 pm on Thursday, October 3rd. Remember to **fill in** all the information above and **staple** all your sheets together, including this one. All exercises are weighed equally unless otherwise stated.

Recollections. Recall that for every $n, k \in \mathbb{N}$ the *binomial number* $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, and that the greatest common divisor of a tuple (a, b) is the unique positive integer d with the property that $d \mid a$, $d \mid b$ and if $d' \mid a$ and $d' \mid b$ then $d' \mid d$.

Exercise 1. Show that for each natural number n, we have that 2 divides $\binom{2n}{n}$, and that $n+1$ also divides $\binom{2n}{n}$.

Exercise 2. Show that for any integer m the greatest common divisor between $5a+8$ and $7a+3$ is either 1 or 41. Find an integer so that it equals 1, and check that for $a = 23$, this number is indeed 41.

Exercise 3. Following the extended Euclidean algorithm, find a pair of integers (x, y) such that $123x + 164y = 41$. Record all steps of the extended algorithm in the form of a table, or otherwise.

Exercise 4.

1. Show by induction that for every $n \in \mathbb{N}$ we have that $1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$.
2. Show by induction that for every $n \in \mathbb{N}$ we have that $1^3 + 2^3 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2$.
3. **Bonus:** show by induction that for every natural number n the sum $1^p + \cdots + n^p$ is a polynomial in n of degree $p+1$.