Assignment 1

Preliminaries

MAU22101 — Group Theory

NAME AND SURNAME: ....

STUDENT NUMBER: .... NUMBER OF PAGES: ....

Note. Solutions to this assignment are **due** by 3:00 pm on Thursday, September 26th. Remember to **fill in** all the information above and **staple** all your sheets together, including this one. All exercises are weighed equally unless otherwise stated.

Recollections. Recall that the *support* of a permutation σ is the set $X(\sigma) := \{i : \sigma(i) \neq i\}$. For example, $X(\mathrm{id}) = \varnothing$, while for $X((12)(34)) = \{1,2,3,4\}$. If σ is a permutation, a *power of* σ is any permutation of the form σ^k for some $k \in \mathbb{N}$. Recall that the *sign* of a permutation σ can be computed as the determinant of the matrix $M(\sigma)$ with $M(\sigma)_{ij} = 0$ unless $j = \sigma(i)$, in which case $M(\sigma)_{ij} = 1$. Alternatively, the sign of σ is 1 if it can be written as a product of an even number of transpositions, and -1 if it can be written as a product of an odd number of transpositions. For example, $\mathrm{sign}(123) = 1$ while $\mathrm{sign}(12) = -1$.

Exercise 1. By considering the cycle (1234) in S_4, show that the power of a cycle may not be a cycle. Compute all three distinct powers of (132) for a non-counterexample.

Exercise 2. Two permutations σ and τ of S_n are called *disjoint* if the sets $X(\sigma)$ and $X(\tau)$ are disjoint. Show that in this case, τ and σ commute, that is, $\tau\sigma = \sigma\tau$.

Exercise 3. Draw a table with the sign of each of the six permutations of S_3. *Hint:* show that the sign of a cycle of even length is odd, and the sign of a cycle of odd length is even.

Exercise 4.

Let $f : X \to Y$ be a function and let $A, A' \subseteq X$ and $B, B' \subseteq Y$. Prove the following:

1. $f\left(f^{-1}(B)\right) \subseteq B$ and $A \subseteq f^{-1}\left(f(A)\right)$.
2. If $B \subseteq B'$, then $f^{-1}(B) \subseteq f^{-1}(B')$, and if $A \subseteq A'$, then $f(A) \subset f(A')$.
3. $f^{-1}\left(B \cup B'\right) = f^{-1}(B) \cup f^{-1}(B')$ and $f^{-1}\left(B \cap B'\right) = f^{-1}(B) \cap f^{-1}(B')$.
4. $f(A \cup A') = f(A) \cup f(A')$ and $f\left(A \cap A'\right) \subseteq f(A) \cap f(A')$, but give a counterexample to show the converse inclusion fails.