The non-commutative calculus of fields and forms through dg-resolutions

Pedro Tamaroff
Trinity College Dublin

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Motivation and origins: the Cartan calculus

For a smooth manifold M, the spaces $\Omega(M)$ of forms on M and $\Theta(M)$ of polyvector fields on M are endowed with a Cartan calculus

Similarly, for a smooth commutative algebra A, we know from the HKR theorem that we have identifications

$$\mathrm{HH}_*(A) = \Lambda_A^* \Omega_A^1, \quad \mathrm{HH}^*(A) = \Lambda_A^* \mathrm{Der}(A).$$

which give us a "Cartan calculus" for A: a wedge product on fields, a contraction of forms with fields, a de Rham differential on forms, and a Lie bracket on fields.

The non-commutative analogue

We can produce an analogous picture when A is an arbitrary associative algebra (Daletski–Gelfand–Tsygan '90), the *Tamarkin–Tsygan calculus of A*, and write it

$$\mathsf{Calc}(A) = (\mathrm{HH}^*(A), \mathrm{HH}_*(A)).$$

This is a pair of the form (V, M) where V is a Gerstenhaber algebra and M is a V-module along with a differential d relating the Lie module and the module structure of M through "Cartan's magic formula":

$$[i, d] = L.$$

Theorem. (Armenta–Keller '18) The Tamarkin–Tsygan calculus of an algebra is derived invariant.

An definition intrinsic to dg resolutions

The above produces an assignment (not a functor) from associative algebras to Tamarkin–Tsygan calculi.

From the work of Jim Stasheff ('93), we know the bracket is "intrinsic" to the homotopy category of dg algebras: we can compute it as the Lie bracket on derivations of any good dg resolution of our algebra.

Question. What about the whole Tamarkin–Tsygan calculus? Can we produce from the homotopy type of A a datum that gives this calculus and from which it can be effectively computed?

From now on, let us fix a dg replacement $(TV, \partial) = B \longrightarrow A$.

Standard resolution

If $TV = B$ is a free algebra, there is a "standard" resolution in ${}_B\mathrm{Mod}_B$

$$\mathrm{St}_*(B) : 0 \longrightarrow B \otimes V \otimes B \longrightarrow B \otimes B \longrightarrow B \longrightarrow 0$$

where, in addition, we have internal differentials coming from ∂.

If $\mathrm{Bar}_*(B)$ is the double-sided bar resolution, there is a retraction of resolutions

$$\pi : \mathrm{Bar}_*(B) \longrightarrow \mathrm{St}_*(B), \quad i : \mathrm{St}_*(B) \longrightarrow \mathrm{Bar}_*(B).$$

where i is the inclusion and π is *very* simple.

Conclusion: we can compute the underlying (co)homology groups of $\mathrm{Calc}(A)$ through the standard resolution $\mathrm{St}_*(B)$.

Non-commutative fields and forms

Note that the complexes $\mathsf{St}_*(B)_B$ and $\mathsf{St}_*(B)^B$ are in fact naturally isomorphic to

$$\mathcal{V}(B) = (\mathrm{ad} : B \longrightarrow \hom(V, B)), \quad \Omega(B) = (\mathrm{co} : B \otimes V \longrightarrow B)$$

respectively, which we call the complexes of non-commutative fields and non-commutative forms on B.

Problem: we can compute the calculus of A through $\mathsf{Bar}_*(B)$, but can we do this with these smaller complexes?

Answer: this depends on how well we understand how calculi behave under retractions!

A structure on Hochschild (co)chains

Deligne's question: can one lift the Gerstenhaber algebra structure on $\mathrm{HH}^*(A)$ to the chain level? Yes, the solution involves formality of the little disks operad.

It is reasonable to consider the same problem for the Tamarkin–Tsygan calculus structure on $\mathsf{Calc}(A)$.

Theorem (Kontsevich-Soibelman) There is a formal geometric operad C that solves Deligne's conjecture for $\mathsf{Calc}(A)$: there is an action of C on the pair $(C^*(A), C_*(A))$ so that taking homology we get the usual calculus.

Homotopy calculi

- Classical structures (commutative, Lie, associative, Gerstenhaber) have "homotopy coherent" versions.
- One can do the same for calculi if one finds a dg replacement of the operad Calc controlling calculi.
- Note this operad admits a quadratic-cubic presentation, owing to the Cartan magic formula.

Theorem (T.) The operad Calc is inhomogeneous Koszul.

It follows that one can consider a reasonable notion of homotopy coherent calculi, and this notion behaves just as good as the classical ones.

Homotopy transfer

To solve our problem above, we put together

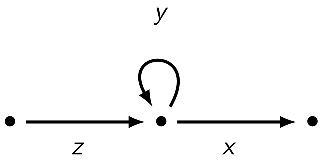
- the result of Kontsevich–Soibelman and
- the dg replacement Calc_∞ of Calc.

Corollary (Daletskii–Tamarkin–Tsygan) For every algebra A, the pair of Hochschild cochains $(C^*(A), C_*(A))$ admits a homotopy coherent calculus structure.

Corollary (T.) The pair $(\mathcal{V}(B), \Omega(B))$ admits a homotopy coherent calculus structure that is equivalent to the homotopy coherent calculus on $(C^*(A), C_*(A))$.

A small quiver

Let us consider the following quiver Q with relations $R = \{xy^2, y^2z\}$. We will compute its minimal dg resolution and with part of its calculus.

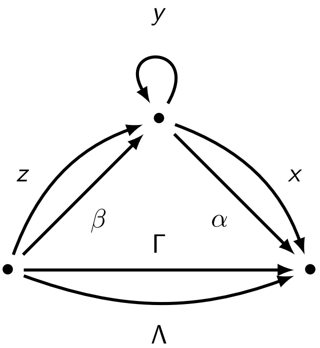


The dg replacement B is given by the free algebra over $\mathbb{k}Q_0$ with set of homogeneous generators $\{x, y, z, \alpha, \beta, \Gamma, \Lambda\}$ such that

$$\partial x = \partial y = \partial z = 0,$$
$$\partial \alpha = xy^2, \quad \partial \beta = y^2 z,$$
$$\partial \Gamma = \alpha z - x\beta, \quad \partial \Lambda = xy\beta - \alpha yz.$$

Quiver of the dg resolution

The (dg) quiver of B looks as follows



and we now consider the complex of nc fields $\mathcal{V}(B) = (B \longrightarrow \mathrm{Der}(B))$ on B to compute $\mathrm{HH}^*(A)$ (one can compute all the calculus with it!).

Computation of $\mathrm{HH}^1(A)$

We can compute the 0-cycles directly:

$$
\begin{array}{llll}
E_s(x) = 0, & E_s(y) = y^{s+1}, & E_s(z) = 0, & E_s(\alpha) = 2\alpha y^s, \\
E_s(\beta) = 2y^s\beta & E_s(\Lambda) = 3\alpha y^{s-1}\beta, & E_s(\Gamma) = -2\alpha y^{s-2}\beta, & \\
F_s(x) = xy^s, & F_s(y) = 0, & F_s(z) = 0, & F_s(\alpha) = \alpha y^s, \\
F_s(\beta) = 0 & F_s(\Lambda) = \alpha y^{s-1}\beta, & F_s(\Gamma) = -\alpha y^{s-2}\beta, & \\
G_s(x) = 0, & G_s(y) = 0, & G_s(z) = y^s z, & G_s(\alpha) = 0, \\
G_s(\beta) = y^s\beta, & G_s(\Lambda) = \alpha y^{s-1}\beta, & G_s(\Gamma) = -\alpha y^{s-2}\beta. &
\end{array}
$$

$\mathrm{HH}^1(A)$ is infinite dimensional with basis the classes of the elements in $\{F_0, G_0, E_s : n \in \mathbb{N}_0\}$. For each $s, t \in \mathbb{N}_0$,

$$[E_s, E_t] = (s-t)E_{s+t}, \quad [F_0, -] = [G_0, -] = 0.$$

We get abelian algebra $\mathbb{k}^2$ acting trivially on the Witt algebra.

Computation of $\mathrm{HH}^2(A)$

The following derivations form a basis of the 1-cycles in $\mathrm{Der}(B)$, where unspecified values are zero, $s \in \mathbb{N}_0$, and we agree that $y^{-1} = y^{-2} = 0$:

$$
\begin{array}{llll}
\Phi_s(\alpha) = xy^s, & \Phi_s(\beta) = y^s z, & \Phi_s(\Lambda) = \alpha y^{s-1} z, & \Phi_s(\Gamma) = -\alpha y^{s-2} z, \\
\Phi'_s(\alpha) = 0, & \Phi'_s(\beta) = y^{s+2} z, & \Phi'_s(\Lambda) = -\alpha y^{s+1} z, & \Phi'_s(\Gamma) = \alpha y^s z, \\
\Pi_s(\alpha) = 0, & \Pi_s(\beta) = y^{s+2}, & \Pi_s(\Lambda) = \alpha y^{s+1}, & \Pi_s(\Gamma) = \alpha y^s, \\
\Pi'_s(\alpha) = xy^s z, & \Pi'_s(\beta) = 0, & \Pi'_s(\Lambda) = 0, & \Pi'_s(\Gamma) = 0, \\
\Psi_s(\alpha) = 0, & \Psi_s(\beta) = y^{s+2} z, & \Psi_s(\Lambda) = -\alpha y^{s+1} z, & \Psi_s(\Gamma) = xy^s \beta, \\
\Theta_s(\alpha) = 0, & \Theta_s(\beta) = 0, & \Theta_s(\Lambda) = \alpha y^s z - xy^s \beta, & \Theta_s(\Gamma) = 0, \\
\Xi_s(\alpha) = 0, & \Xi_s(\beta) = 0, & \Xi_s(\Lambda) = 0, & \Xi_s(\Gamma) = \Theta_s(\Lambda).
\end{array}
$$

It turns out a basis of $H^1(\mathrm{Der}(B))$ is given by the classes of the derivations Φ_0, Φ_1 so that $\mathrm{HH}^2(A)$ is two dimensional.

Computation of $\mathrm{HH}^3(A)$ and the bracket

A basis for the 2-cycles is given by the following family of derivations, where $s \in \mathbb{N}_0$ and $t \in \{0, 1\}$:

$$\Omega_s^t(\Lambda) = 0, \qquad \Omega_s^t(\Gamma) = xy^s z^t, \qquad \Upsilon_s^t(\Lambda) = xy^s z^t, \qquad \Upsilon_s^t(\Gamma) = 0.$$

It is straightforward to check that all of these are boundaries except for Υ_0^1 and Υ_0^0. The bracket is as follows:

$$\begin{aligned}
&[E_{s+2}, \Phi_t] = 3\Xi_{s+t+1} - 2\Theta_{s+t}, && [F_{s+2}, \Phi_t] = \Theta_{t+s+1} - \Xi_{t+s},\\
&[G_{s+2}, \Phi_t] = \Theta_{t+s+2} - \Xi_{t+s+2}, && [F_1, \Phi_t] = [G_1, \Phi_t] = \Theta_t,\\
&[E_s, \Upsilon_t^r] = (t - 3\delta_{s,0})\Upsilon_{s+t}^r. && [E_s, \Omega_t^r] = (t + 2\delta_{s,0})\Omega_{s+t}^r,\\
&[F_0, -] = [G_0, -] = 2 && \text{on } \langle \Omega_s^t, \Omega'^t_s : s \in \mathbb{N}_0 \rangle,\\
&[F_0, -] = [G_0, -] = [E_0, -] = 0 && \text{on } \langle \Upsilon_s^t, \Upsilon'^t_s, \Phi_s : s \in \mathbb{N}_0 \rangle,\\
&[E_1, \Phi_t] = 3\Xi_t.
\end{aligned}$$

Thank you!

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