The pigeon-hole principle and double counting

Pedro Tamaroff

12th September 2018

1 Introduction

1.1. The pigeon-hole principle is an elementary result from enumerative combinatorics, as is the principle of double counting, and both can be labeled a triviality. However, echoing George Bergman [2], we recall that

> *"…what is trivial when described in the abstract can be far from clear in the context of a complicated situation where it is needed."*

1.2. The purpose of what follows is to show, through a few concrete examples, that these two principles can be used to prove remarkable and non-trivial results. We will being with rather simple examples, and work our way up to prove Sperner's lemma in two dimensions. It is worth to remark that one can extend the lemma to arbitrary dimensions, and that it can be used, as originally intended by Sperner, to give a proof of the Brouwer fixed point theorem, and thus a proof of the theorem of invariance of domain. We will not pursue this in the notes, but do recommend the reader to read about this on their own.

2 The pigeon–hole principle

2.1. Let us begin by stating the pigeon-hole principle: if n pigeons fly into r pigeon-holes and $n > r$, then at least one pigeonhole will contain two pigeons. We can be slightly more precise by noting that in fact at least one pigeonhole will contain $\lceil n/r \rceil$ pigeons. As we warned the reader in the introduction, the proof of this is easy. Indeed, if every pigeon-hole contained less than n/r pigeons, the total tally of pigeons would be less than $r \cdot n/r = n$, which cannot be.

2.2. We now prove the following claim using the pigeon-hole principle.

Claim. *Consider the numbers* $1,2,3,\ldots,2n$*, and pick any* $n+1$ *of them. Then there are two which are relatively prime, and there are also two such that one divides the other.*

Proof. For the first claim we note that, among $n+1$ numbers obtained from a list of $2n$ consecutive numbers, there must be two, one which is the successor of the other. Indeed, if this was not the case, this set of $n+1$ numbers would not be contained in $\{1,\ldots,2n\}$. Since k and $k+1$ are always coprime, we obtain what we want.

The second claim is slightly less obvious. We can write every number in the list we chose in the form $2^k m$, where m is odd. Since our list comes from the set $\{1,\ldots,2n\}$, the odd number m is in the set $\{1,3,\ldots,2n-1\}$, which has only n elements. The pigeon-hole principle then says there must be two elements in our set of $n+1$ elements which have the same odd part, and this gives what we wanted. ◀

2.3. To conclude this section, we consider the following result, which is a not-so-trivial example of a "Ramsey problem". Broadly speaking, Ramsey theory tells us that large structures of seemingly arbitrary shape must contain certain ordered substructures. Our example deals with numbers and monotone subsequences.

Claim. *Any sequence of* $mn+1$ *distinct real numbers contains either an increasing subsequence of length* $m+1$*, or a decreasing sequence of length* $n+1$*, or both.*

2.4. By *increasing* or *decreasing* we mean the subsequence is monotone and respects the original enumeration of the sequence. For example, if our sequence is $(1,4,3,5,6)$, of length $5 = 2\cdot 2+1$, $(1,3,6)$ is an increasing subsequence of length 3, but there are no decreasing subsequences of this same length.

Proof. Let us consider, as in the statement of the claim, $nm+1$ distinct ordered real numbers $X = \{x_1,\ldots,x_{nm+1}\}$. We define a function $f : X \to \mathbb{N}$ which assigns to x_i the length $f(i)$ of the longest *increasing* subsequence starting at this number. Observe that if for some i we have that $f(i) \geqslant m+1$, we have an increasing subsequence of length $m+1$, so we are done. We can then assume that f has image in $\{1,\ldots,m\}$, and obtain $f : X \to \{1,\ldots,m\}$.

Since $|X| = mn+1$, the pigeon-hole principle says that there must be some $s \in \{1,\ldots,m\}$ so that $f(x_{i_j}) = s$ for some choice of $\lceil \frac{mn+1}{m} \rceil = n+1$ subindices $i_1 < \cdots < i_{n+1}$. To conclude our proof, we observe that if we had $x_{i_j} < x_{i_{j+1}}$, this would mean $f(i_{j+1}) \geqslant s+1$, which cannot be. It follows that the subsequence we have picked is *decreasing*, which is what we wanted. ◀

3 Double counting

3.1. The principle of double counting, which we can also call the principle of "counting things in two ways", is as follows. Suppose A and B are finite sets, and choose a subset S of $A \times B$. Let us say that $a \in A$ is incident to $b \in B$ exactly when $(a, b) \in S$. If B_a is the number of elements of B incident to $a \in A$, and A_b the number of elements of A incident to $b \in B$, then

$$\sum_{a \in A} B_a = |S| = \sum_{b \in B} A_b.$$

3.2. If we set up a matrix M of zeros and ones, indexed by $A \times B$, so that $M_{(a,b)} = 1$ if $(a, b) \in S$ and $M_{(a,b)} = 0$ if not, then for $b \in B$, A_b is the sum of the b-th column, while for $a \in A$, B_a is the sum o the a-th row. The principle of double counting says that, naturally, the sum of all entries of M, which is just the cardinality of S, can be obtained by either adding up elements in rows first and then adding them up, or by either doing this for the columns and then, again, adding them up.

3.3. Recall that a simple undirected graph G is the data of a set of vertices V and a set of edges E, given by a subset of 2-element subsets of V. Thus, there is an edge between vertex v and w in V exactly when $\{v, w\} \in E$. The *degree* of a vertex is the number of edges it belongs to, and we write it $\deg(v)$. Double counting gives us the following result, which implies in particular the so-called *handshaking lemma*.

Claim. *The following* degree-sum formula *holds for every finite simple undirected graph G:*

$$\sum_{v \in V} \deg(v) = 2|E|.$$

In particular, the number of vertices of odd degree of any finite simple undirected graph must be even.

Proof. On the set $V \times E$, we consider the subset (v, e) where v is a vertex in the edge e. Counting edges, this set has $2|E|$ elements, since every edge contains exactly two endpoints. Counting vertices we obtain the left hand side of the degree-sum formula, since each vertex is incident to exactly $\deg(v)$ edges. To prove the final remark in the claim, we reduce the degree-sum formula modulo two, noting that the left hand side will reduce to $\deg v$ to 1 whenever $\deg v$ is odd, while vertices of even degree will not contribute. ◄

4 The Sperner lemma

4.1. To conclude these set of notes, we state and prove the celebrated Sperner lemma in two dimensions, starting with some geometrical considerations. A *triangulation* of a triangle is a decomposition of it into smaller triangles which fit together edge-by-edge. Let us take our triangle to be the 2-simplex Δ^2, and call its vertices v_0, v_1, v_2.

4.2. A *Sperner colouring* of a triangulation T of Δ^2 is obtained by following these rules:

(1) Color v_0 blue, v_1 red and v_2 yellow.
(2) Any vertex in the line joining v_0 with v_1 is colored blue or red, any one in the line joining v_1 and v_2 red or yellow, and any in the line joining v_2 and v_0 yellow or blue.
(3) Finally, any vertex in the interior of the simplex is coloured arbitrarily with either of the three colours.

Lemma (Sperner, 1928)**.** *Any Sperner colouring of a triangulation of the* 2*-simplex contains a triangle whose vertices have different colours.*

Proof. On every edge which has one vertex painted red and one blue, we will draw a door. In particular, on the line segment from v_0 to v_1, where we go from colour blue to red, there must be an odd number of opened doors. Through each door, enter the triangulation, and observe that at each step, exactly two options occur: either the triangle is tri-coloured, and we have reached a dead end, or the vertex opposite to the door is either blue or red and we can move on. Once we have done this with every door, there are a number of dead end paths and a number of paths which lead us in and out of our triangulation. To conclude our proof, we note that the in and out paths pair up an even number of our doors, so there must be at least one dead end door, and hence one tri-coloured triangle. ◀

4.3. The reader may have noticed that we did not use any of the techniques developed in the previous sections to prove the Sperner lemma. Is this *really* the case?

References

[1] Martin Aigner and Günter M. Ziegler, *Proofs from The Book*, 4th ed., Springer-Verlag, Berlin, 2010. MR2569612

[2] George M. Bergman, *The diamond lemma for ring theory*, Adv. in Math. **29** (1978), no. 2, 178–218, DOI 10.1016/0001-8708(78)90010-5. MR506890