Cohomology of combinatorial species

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Goals

- Study the category ${}_{\Bbbk}\mathrm{Sp}$ of combinatorial species with its monoidal structure given by the *Cauchy product*.
- Study its coalgebras, in particular the *exponential species* **e**: a coalgebra in ${}_{\Bbbk}\mathrm{Sp}$ is the same as an **e**-bicomodule.
- We get a functor

$$\mathbf{x} \text{ coalgebra in } {}_{\Bbbk}\mathrm{Sp} \longmapsto H^*(\mathbf{x}) = \mathrm{Ext}^*_{\mathbf{e}\text{-}\mathbf{e}}(\mathbf{x}, \mathbf{e})$$

which we want to understand as completely as possible.

Combinatorial species

Fix a commutative ring $\Bbbk$. A *combinatorial species over* $\Bbbk$ is a functor

$$\mathbf{x} : \mathrm{FinSet}^{\times} \longrightarrow {}_{\Bbbk}\mathrm{Mod}.$$

Such functors are the objects of a $\Bbbk$-linear abelian category ${}_{\Bbbk}\mathrm{Sp}$.

If $\mathbf{x}$ and $\mathbf{y}$ are species, their Cauchy product is such that

$$(\mathbf{x} \otimes \mathbf{y})(I) = \bigoplus_{S \sqcup T = I} \mathbf{x}(S) \otimes_{\Bbbk} \mathbf{y}(T)$$

Along with more data, this endows the category ${}_{\Bbbk}\mathrm{Sp}$ with a symmetric monoidal structure.

The upshot of a monoidal structure

Coalgebas, algebras and Hopf algebras in $_{\Bbbk}\mathrm{Sp}$ codify useful and classical combinatorial operations of "restriction" and "gluing" of structures.

Example: if **P** is the species of posets, there is $\mu : \mathbf{P} \otimes \mathbf{P} \longrightarrow \mathbf{P}$ such that

$$\mu(p^1 \otimes p^2) = p^1 * p^2 \qquad \text{(ordinal sum)}$$

Example: if **G** is the species graphs, there is $\Delta : \mathbf{G} \longrightarrow \mathbf{G} \otimes \mathbf{G}$ such that

$$\Delta(g) = \sum g_S \otimes g_T \qquad \text{(induced subgraphs)}$$

The exponential species

The exponential species $\mathbf{e} : \mathsf{Set}^{\times} \longrightarrow {}_{\Bbbk}\mathsf{Sp}$ is such that $\mathbf{e}(I) = \Bbbk\{e_I\}$ for each finite set I. It is a bialgebra, with

$$e_S e_T = e_{S \cup T}, \qquad \Delta(e_I) = \sum e_S \otimes e_T$$

The category of "linear" $\mathbf{e}$-bicomodules is equivalent to that of "linear" coalgebras.

Cohomology

To each linear coalgebra $\mathbf{x}$ in ${}_{\Bbbk}\mathsf{Sp}$ we associate cohomology groups

$$H^*(\mathbf{x}) = \operatorname{Ext}^*_{\mathbf{e}\text{-}\mathbf{e}}(\mathbf{x}, \mathbf{e}).$$

We can compute $H^*(\mathbf{x})$ with a canonical complex that resolves $\mathbf{e}$ by applying $\hom_{\mathbf{e}\text{-}\mathbf{e}}(\mathbf{x}, ?)$ to

$$\Omega\mathbf{e} : 0 \longrightarrow \mathbf{e}^{\otimes 2} \longrightarrow \mathbf{e}^{\otimes 3} \longrightarrow \mathbf{e}^{\otimes 4} \longrightarrow \cdots$$

Problem: although theoretically useful, this complex is computationally ineffective.

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Concrete computations: linear orders

Let $\mathbf{L} : \mathsf{Set}^{\times} \longrightarrow {}_{\Bbbk}\mathsf{Mod}$ be the species of linear orders. This is the free noncommutative algebra on one generator on cardinality 1.

Theorem

If $\Bbbk$ is a PID and if $\mathbb{Q} \subseteq \mathsf{Frac}(\Bbbk)$, there is an algebra isomorphism

$$H^*(\mathbf{L}) \simeq \Bbbk[\xi, \eta]$$

with $|\eta| = 1$ and $|\xi| = 2$.

In particular $H^n(\mathbf{L})$ is one dimensional for each $n \in \mathbb{N}_0$.

Concrete computations: simplicial complexes

We can associate to every finite simplicial complex K a left **e**-comodule $\mathbf{x}_K$.

Theorem

If $\Bbbk$ is a PID, there are isomorphisms

$$H^*(\mathbf{x}_K^t) \simeq H^*(K)^{[-1]} \qquad H^*(\mathbf{x}_K^s) \simeq SR(K)$$

where $SR(K)$ is the graded Stanley-Reisner ring associated to K.

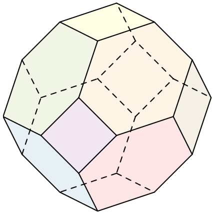
Example: if $K = \partial\Delta^2$, then

$$SR(K) = \Lambda[x, y, z]/(xyz)$$

The Coxeter complex and the braid arrangement

Key step: use the explicit description of the S-module structure of the cohomology groups of the Coxeter complexes of type A. Their complexes of simplicial chains appear naturally as the sequence

$$\left\{ 0 \longrightarrow \mathbf{e}^{\otimes 0}(n) \longrightarrow \mathbf{e}^{\otimes 1}(n) \longrightarrow \mathbf{e}^{\otimes 2}(n) \longrightarrow \cdots \right\}_{n \geqslant 0}$$



The combinatorial complex

We were able to abstract a general method from our computations.

Theorem I

Let $\mathbf{x}$ be an $\mathbf{e}$-bicomodule. There is a cochain complex $CC^*(\mathbf{x})$ with

$$CC^n(\mathbf{x}) = \hom_{S_n}(\mathbf{x}(n), \Bbbk(n)^-)$$

so that

$$H(CC^*(\mathbf{x})) \simeq H^*(\mathbf{x})$$

if $\mathbf{x}(n)$ is $\Bbbk[S_n]$-projective for each $n \in \mathbb{N}_0$.

Moreover, if $\mathbf{x}$ is a coalgebra, $CC^*(\mathbf{x})$ is a DGA algebra, and the isomorphism is one of DGA algebras.

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Results

With the combinatorial complex we were able to

1. Give an alternative description of $H^*(\mathbf{x})$.
2. Provide a computationally simple formula for the product in $H^*(\mathbf{x})$.
3. Deduce structural results:
    - a Künneth formula,
    - graded commutativity for commutative coalgebras,
    - vanishing of cohomology groups

Example:

- Theorem I gives that $CC^n(\mathbf{L}) \simeq \Bbbk$ for each $n \in \mathbb{N}_0$,
- As $\mathbf{L}$ es commutative, by Theorem II we have $d = 0$.
- Thus $H^n(\mathbf{L}) \simeq \Bbbk$ for each $n \in \mathbb{N}_0$, which recovers our computation.

More generally: if $S_{\mathbf{x}} = \mathbf{e} \circ \mathbf{x}$ is the free commutative coalgebra on a positive species $\mathbf{x}$, $d = 0$ and the combinatorial complex *is* the cohomology algebra of $S_{\mathbf{x}}$.

The differential

Theorem II

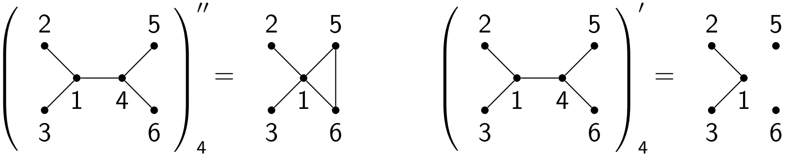
If $\Bbbk$ is a field of characteristic zero, the differential d of $CC^*(\mathbf{x})$ is such that, for each $f \in CC^n(\mathbf{x})$ and $z \in \mathbf{x}(n+1)$,

$$d(f)(z) = \sum_{i=1}^{n+1} (-1)^{i-1} \left(f(z_i') - f(z_i'') \right).$$

In particular, $d = 0$ if $\mathbf{x}$ is a cosymmetric bicomodule.

Some examples

Graphs: we can endow the species of graphs **G** with a non-commutative comultiplication. With this structure:



The spectral sequence

The argument that gives rise to $CC^*(\mathbf{x})$ can be generalized to (attempt to) compute $\mathrm{Ext}^*_{\mathbf{h}\text{-}\mathbf{h}}(\mathbf{x},\mathbf{h})$ with $\mathbf{h}$ a coalgebra in ${}_{\mathbb{k}}\mathrm{Sp}$.

Theorem IV

There is a spectral sequence with

$$E_1^{p,q} \simeq \hom_{S_p}(\mathbf{x}(p), \mathrm{H}^{p,q}(\mathbf{h})) \underset{p}{\Longrightarrow} \mathrm{Ext}^{p+q}_{\mathbf{h}\text{-}\mathbf{h}}(\mathbf{x},\mathbf{h}),$$

where $\mathrm{H}^{p,q}(\mathbf{h}) = \mathrm{Cotor}^{p,p-q}_{\mathbf{h}}(\mathbb{k},\mathbb{k})$

Computing Cotor

Computing these Cotor groups is a complex combinatorial problem —this includes understanding them as $\mathfrak{S}$-modules! We did this for **e**, where we know

$$\operatorname{Cotor}_{\mathbf{e}}^{p,q}(\mathbb{k},\mathbb{k}) = \begin{cases} \mathbb{k}^{-}(p) & \text{if } p = q, \\ 0 & \text{else.} \end{cases}$$

Problem: for the species **L** of linear orders, with the aid of a computer, we obtained that for $0 \leqslant p, q \leqslant 5$

$$\dim_{\mathbb{k}} \operatorname{Cotor}_{\mathbf{L}}^{p,q}(\mathbb{k},\mathbb{k}) = s(p,q),$$

where the numbers $s(n,k)$ are the Stirling numbers of the first kind. We can prove this for entries (p,p), where we know $\operatorname{Cotor}_{\mathbf{L}}^{p,p}(\mathbb{k},\mathbb{k})$ is the sign representation of S_p. What about the others?

Questions?

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