Minimal models for monomial algebras

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Resolutions of objects

What we do in the shadows:

- Generally, we want to study an object Y which is linear in nature, for example, a module over an algebra.
- A resolution of Y is, loosely speaking, a way to unpack Y into a sequence of objects which are much more well behaved than Y.
- We relate these objects by a sequence of *boundary maps*. These encode the "homological properties" of Y in a non-trivial way.

Some examples

- We can resolve modules over algebras by free modules. Free modules behave like vector spaces, and this makes us happy. We can do the same with bimodules.
- We can resolve algebras by *differential graded algebras* that are free as an algebra. Free algebras depend only on a vector space, and this also makes us happy.

Notation. We will write (TX, d_X) for a free algebra with a differential, and call a *resolution of an algebra* a *model*.

Why do we like models?

- Resolving an algebra resolves, in particular, its representations: we can focus on understanding how to obtain models.
- Models are algebras, which are non-linear in nature, so they capture more information than representations, which are linear.
- They have been used to solve open problems, like Han's conjecture [2] and the rational Ganea conjecture [3].

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Associative algebras

Associative algebras can be presented by *generators* V and *relations* R. Resolving an algebra involves understanding its syzygyes.

- The *free algebra* has no relations.
- The *polynomial algebra* has relations $R = \{xy - yx \mid x, y \in V\}$.
- The *exterior algebra* has relations $R = \{xy + yx \mid x, y \in V\}$.
- A *monomial algebra* has relations that are monomials on V, like $xy^2 = 0$ or $xyx = 0$.

Notation. We will write $A = T\langle V | R\rangle$ for an algebra with generators V and relations R, and $\overline{A}$ for the *augmentation ideal*.

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The plan

Principle of conservation of difficulty. We want to *resolve an algebra* by some object (TX, d_X). In general, if we can pin down one of X or d_X easily, a lot of work will be needed to pin down the other.

- Show that X can be described combinatorially in terms of *Anick chains*. For example, $X_0 = V$, $X_1 = R$.
- Produce a datum that remembers how to compute X from a "canonical model" of A.
- Show that this datum determines d_X and then the desired model (TX, d_X) of A.
- Give explicit formulas for d_X on generators.

Anick chains: combinatorics of syzygyes

If $A = T\langle V|R\rangle$ is an associative *monomial* algebra, let us write $X_0 = V$ for its generators and $X_1 = R$ for its relations.

Inductively define $X_2, X_3, \ldots$ by their bases $C_2, C_3, \ldots$, consisting of monomials, called chains, as follows.

Definition. A monomial m is an i-chain if the following holds:

- We can write $m = m_1 m_2 = m_1' m_2'$ so that,
- m_1 is an $(i-1)$-chain and m_2' is a 1-chain,
- m_2 is a *proper* right divisor of m_2' and,
- no proper left divisor of m can be written so to satisfy the previous conditions.

Anick chains: useful examples

Example 1. Let us consider $\Bbbk\langle t|t^2\rangle$. Then $C_0 = \{t\}$, $C_1 = \{t^2\}$, and in general $C_n = \{t^{n+1}\}$ for every $n \in \mathbb{N}_0$.

Example 2. Let us consider $\Bbbk\langle x, y|x^2, xy^2\rangle$. Then $C_n = \{x^{n+1}, x^n y^2\}$ for each $n \in \mathbb{N}$, while $C_0 = \{x, y\}$.

Example 3. Let us consider $\Bbbk\langle x, y, z|xy^2, y^2z\rangle$. Then $C_0 = \{x, y, z\}$, $C_1 = \{xy^2, y^2z\}$, $C_2 = \{xy^2z, xy^3z\}$, and $C_n = 0$ for $n \geqslant 3$.

The bar construction: syzygyes are homological

An important construction that can be extracted from an algebra A is the *bar construction of* A, (BA, d_{BA}, Δ). It encodes syzygyes of A.

- We let BA be the free coalgebra on $\overline{A}$, so it is a sum $\mathbb{k}, \overline{A}, \overline{A}^{\otimes 2}, \ldots$.
- The operator d is induced by the multiplication of A:

$$d_{BA}[a_1|\cdots|a_n] = \sum_{i=1}^{n-1}(-1)^{i-1}[a_1|\cdots|a_{i-1}|a_i a_{i+1}|a_{i+2}|\cdots|a_n]$$

- The operator $\Delta : BA \to BA \otimes BA$ is *deconcatenation*:

$$\Delta[a_1|\cdots|a_n] = \sum_{i=1}^{n}[a_1|\cdots|a_i] \otimes [a_{i+1}|\cdots|a_n]$$

The differential of BA

One can check that $d_{BA}^2 = 0$ is *equivalent* to the fact A is *associative*. The operator Δ is coassociative and makes BA into a *dg coalgebra*. We write Tor^A for its *homology*.

Theorem (Anick and Green–Happel–Zacharia)

Chains give a basis of Tor^A for A a monomial algebra: for each $r \in \mathbb{N}_0$, we have that $\operatorname{Tor}^A_{r+1} \simeq X_r$.

Note! There is a shift in the indices between X and Tor^A. We will denote this by saying that $X = s^{-1}\operatorname{Tor}^A$.

What do we have so far?

The spaces of chains X produced for us the *generators* of our model. But we also need an operator d_X. How can this be obtained from A?

Answer. The space X was obtained from (BA, Δ, d_{BA}), which is a *dg coalgebra*. It turns out that if we relate X and BA by some linear maps, we can build the new operator d_X from this information.

Algebraic discrete Morse theory

Algebraic discrete Morse theory, developed by by Sköldberg [6] and Jöllenbeck–Welker [4], solves our problem. It "remembers" how we went from BA to $H(BA) = \operatorname{Tor}^A$, its homology.

Geometrically, it tells us how to "kill contractible complexes" in BA to obtain a model whose generators are $H(BA)$. The picture we want is:

$$\operatorname{Tor}^A \underset{p}{\overset{i}{\rightleftarrows}} BA \circlearrowright h \quad ,1 - ip = dh + hd, \quad pi = 1$$

...and what it can do for you

Theorem (Jöllenbeck–Welker)

If B is a complex of free modules and M is a Morse matching, there is a complex B^M and maps

$$i : B^M \to B, \quad p : B \to B^M$$

such that $pi = 1_{B^M}$ and ip is homotopic to the identity of B.

In [4], the authors give a Morse matching M on BA so that $(BA)^M$ has basis the set of Anick chains. This gives us the diagram we wanted.

Interlude: dg algebras

We can enhance associative algebras by introducing a grading and an operator on them. We care only about free algebras, say TX.

Grading. We will assume X is a direct sum of $\{X_i\}$ of subspaces. We call X_i the ith graded piece of X.

Differential. We assume that we have an operator $d : TX \to TX$ of degree -1 such that $d^2 = 0$, which satisfies the Leibniz rule

$$d(xy) = dx \cdot y + (-1)^{|x|} x \cdot dy.$$

This rule means d is determined *completely* by its action on X!

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What is a model of an algebra?

Given some dg algebra (TX, d), the 0th homology $H_0(TX)$ is a an associative algebra. We want to exhibit associative algebras in the form $H_0(TX)$ where we *additionally* require that $H_+(TX) = 0$.

Explicitly, given an algebra $T\langle V|R\rangle$, we want to give:

- a graded vector space X,
- a map $f : TX \to A$ and
- a differential d on TX so that

the induced map $Hf : H(TX, d) \to A$ is an *isomorphism*.

...so what do we do with the retraction?

Relevant data. For our purposes, it suffices to understand here

- The homotopy $h : BA \longrightarrow BA$ afforded by ADMT.
- Its behaviour with respect to $\Delta : BA \longrightarrow BA \otimes BA$.
- How these two operators act on Tor_A.

Indeed, the following result describes the differential in the minimal model with generators Tor_A in terms of these two operators.

Transferring the structure to chains

Homotopy Transfer Theorem

Let C be a dga coalgebra (ex. $C = BA$) and consider a retraction (i, p, h) to a complex $p : C \to C'$ as before. Then there is a retraction (j, q, k) between dg algebras $q : T(s^{-1}C) \to T(s^{-1}C)$.

This gives us what we want! Indeed, $T(s^{-1}BA)$ is a (very big) model of A, but now we have a new model, TX, where $X = s^{-1}\operatorname{Tor}^A$ is as small as can be.

How to pin down the differential? Trees.

For a given planar binary tree T, let Δ_T be the operator obtained by decorating the root of T by i (inclusion $\mathrm{Tor}_A \subseteq BA$), its leaves by p (projection $BA \to Tor_A$), its internal edges by h, and its vertices by Δ. These define maps from X to TX.

Theorem (Markl [5] and many others before...)

We have that $d|_{sX} = \sum_{n\geqslant 2} \Delta_n$ where for each $n \in \mathbb{N}_0$, $\Delta_n = \sum_T (-1)^{\vartheta(T)} \Delta_T$ as T ranges through all planar binary trees with n leaves.

Some trees go away. The comb remains.

The explicit description of the differential in TX was simplified with the following vanishing and exchange rules:

Proposition (T.)

If h is the homotopy $h : BA \longrightarrow BA$ and Δ is the (deconcatenation) of BA, then

$$\Delta h = (h \otimes 1)\Delta \mod (\mathrm{Tor}_A \otimes BA).$$

Moreover, h vanishes on Tor_A, and $h^2 = 0$. All this implies that $\Delta_T = 0$ unless T is a *right comb*.

The last of the trees, and its action.

Decompositions. If γ is an r-chain in Tor_A, a *decomposition* of γ is a sequence $(\gamma_1, \ldots, \gamma_n)$ of chains of length $r_1, \ldots, r_n$ such that

- $r_1 + \cdots + r_n = r - 1$ (correct degree),
- $\gamma = \gamma_1 \ldots \gamma_n$ (as monomials).

We say the decomposition has length n, and call it an n-decomposition.

Example. The 2-chain xy^2z has two 2-decompositions, (x, y^2z) and (xy^2, z), and no other decompositions. The 1-chain xy^2 has a single 3-decomposition, (x, y, y), and no other decompositions.

Theorem (T.)

If T is the right comb and γ is any chain, then $\Delta_T(\gamma)$ is a signed sum through all decompositions of γ.

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Description of the model

The end result is the following description of minimal models of monomial quiver algebras.

Theorem (T.)

For each monomial algebra A there is a minimal model $(TX, d_X) \longrightarrow A$. For a chain $\gamma \in X$,

$$d_X(\gamma) = \sum_{n \geqslant 2} (-1)^{\binom{n+1}{2} + |\gamma_1| - 1} \gamma_1 \cdots \gamma_n,$$

where the sum ranges through all possible decompositions of γ.

A non-monomial example

The *super Jordan plane* is the associative algebra

$$\Bbbk\langle x, y | x^2, x^2y - yx^2 - xyx\rangle.$$

Its minimal model has two generators x_n, y_n in degree n for each $n \in \mathbb{N}_0$ corresponding to the ambiguities x^{n+1} and y^2x^n. Its differential vanishes on degree 0 and for $n \in \mathbb{N}_0$ is as follows:

$$dy_{n+1} = y^2x_n - x_ny^2 - \sum_{s+t=n} x_s y x_t - \sum_{\substack{s+t=n \\ t \geqslant 1}} (x_s y_t - (-1)^t y_t x_s),$$

$$dx_{n+1} = \sum_{s+t=n} (-1)^s x_s x_t.$$

Application: the Gerstenhaber bracket

If $TX \to A$ is a model of an algebra A, one can compute Hochschild cohomology of A through derivations of TX.

Fact. The Gerstenhaber bracket in Hochschild cohomology is just the Lie bracket on derivations. So one can compute it easily if one can compute $\mathrm{HH}^*(A)$ through a model. Of course, computing models is quite a task.

One can also compute Hochschild homology and cyclic homology, and other gadgets associated to these invariants.

Thank you!

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