**MA 2331 Homework 1, Solutions**

1. Find the Fourier series representation of the sawtooth function $f$ defined by $f(x) = x$ for $0 \leq x < \pi$, $f(x) = 2\pi - x$, $\pi < x < 2\pi$ and $f(x + 2\pi) = f(x)$.

   *Solution:* $f$ is even, so $b_n = 0$. Also,

$$a_0 = \frac{2}{2\pi} \int_0^{2\pi} dx\ f(x) = \frac{2}{\pi} \int_0^{\pi} dx\ x = \pi$$

   and

$$a_n = \frac{2}{\pi} \int_0^{\pi} dx\ x\ \cos(nx) = \frac{2}{\pi n^2}[(-1)^n - 1]$$

   where we used integration by parts to compute the integral.

2. Compute

$$\int_{-\pi}^{\pi} dx\ \sin mx\ \sin nx$$

   and

$$\int_{-\pi}^{\pi} dx\ \cos mx\ \cos nx$$

   for all integer $m$ and $n$.

   *Solution:* Using the identity

$$2 \sin A \sin B = \cos(A - B) - \cos(A + B),$$

   we get

$$\int_{-\pi}^{\pi} dx\ \sin mx \sin nx = \frac{1}{2} \int_{-\pi}^{\pi} dx\ [\cos(m - n)x - \cos(m + n)x] = \pi\,(\delta_{m,n} - \delta_{m,-n})$$

   Using the identity

$$2 \cos A \cos B = \cos(A + B) + \cos(A - B).$$

   we get

$$\int_{-\pi}^{\pi} dx\ \cos mx \cos nx = \frac{1}{2} \int_{-\pi}^{\pi} dx\ [\cos(m - n)x + \cos(m + n)x] = \pi\,(\delta_{m,n} + \delta_{m,-n})$$

3. The periodic function $f$ is defined by

$$f(x) = \begin{cases} \sin x & 0 < x < \pi \\ 0 & -\pi < x < 0 \end{cases}$$

   and $f(x + 2\pi) = f(x)$.

(a) Represent $f(x)$ as a Fourier series.

*Solution:*This function is neither odd nor even, though the only non-zero $b_n$ coefficient is $b_1 = \frac{1}{2}$ (since $f(x) = \frac{1}{2}\sin x + |\sin x|$ and $|\sin x|$ is even). Now to the $a_n$ coefficients

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dx\ \cos nx\ f(x) = \frac{1}{\pi}\int_0^{\pi} dx\ \cos nx\ \sin x$$

This can be computed via complex exponentials or through the identity $2\sin A\cos B = \sin(A+B) + \sin(A-B)$:

$$a_n = \frac{1}{2\pi}\int_0^{\pi} dx\ [\sin(1+n)x + \sin(1-n)x] = -\frac{1}{2\pi}\left(\frac{\cos(1+n)x}{1+n} + \frac{\cos(1-n)x}{1-n}\right)\Big|_0^{\pi}.$$

Now $\cos(1+n)\pi = \cos(1-n)\pi = -(-1)^n$, and so

$$a_n = -\frac{1}{2\pi}(-(-1)^n - 1)\left(\frac{1}{1+n} + \frac{1}{1+n}\right) = \frac{1}{\pi}(1+(-1)^n)\frac{1}{1-n^2}.$$

This is ambiguous for $n = 1$; it is trivial to check that $a_1 = 0$. Putting everything together

$$f(x) = \frac{1}{\pi} + \frac{2}{\pi}\sum_{n>0,\text{even}} \frac{\cos nx}{1-n^2} + \frac{1}{2}\sin x,$$

or

$$f(x) = \frac{1}{\pi} + \frac{2}{\pi}\sum_{m=1}^{\infty} \frac{\cos 2mx}{1-4m^2} + \frac{1}{2}\sin x.$$

(b) Derive the remarkable formula

$$\frac{1}{2^2-1} + \frac{1}{4^2-1} + \frac{1}{6^2-1} + ... = \frac{1}{2}.$$

*Solution:*$f(0) = 0$ leads to the amazing formula

$$\frac{1}{2^2-1} + \frac{1}{4^2-1} + \frac{1}{6^2-1} + ... = \frac{1}{2}.$$

4. Find the half-range Fourier **sine (odd)** expansion of the function $f(x) = 1$ defined on $0 < x < \pi$.

*Solution:*We have

$$b_n = \frac{1}{\pi}2\int_0^{\pi} \sin(nx) = \frac{2}{n\pi}(1-(-1)^n)$$

so

$$f(x) = \frac{4}{\pi}\sum_{n\ odd} \frac{1}{n}\sin(nx)$$