MA 2331 Homework 2 (to be handed in at 4PM Tuesday October 24 in class)

All questions have the same weight.

1. Express the following periodic functions ($l = 2\pi$) as complex Fourier series

   (a)
   $$f(x) = \begin{cases} 0 & -\pi < x < -a \\ 1 & -a < x < a \\ 0 & a < x < \pi \end{cases}$$
   where $a \in (0, \pi)$ is a constant.

   (b)
   $$f(x) = \frac{1}{2 - e^{ix}}.$$

2. The Riemann zeta function is defined as follows
   $$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad (s > 1).$$

   (a) By applying Parseval's theorem for Fourier series to the sawtooth $f(x) = x$ for $-\pi < x < \pi$ compute
   $$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

   (b) Consider the Fourier expansion of $f(x) = x^2, \quad -\pi < x < \pi$, and use the result to show that
   $$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

3. (a) Compute c_n for $f(x) = e^{\alpha x}, |x| \leq \pi$ and $f(x + 2\pi) = f(x)$. Assume α is a real number.

   (b) Use your result from (a) to compute
   $$\sum_{n=1}^{\infty} \frac{(-1)^n}{\alpha^2 + n^2}$$

4. (a) Determine the Fourier transform of the Gaussian function
   $$f(x) = e^{-\alpha x^2},$$
   where α is a positive constant.

   (b) Compute
   $$e^{-\alpha x^2} * e^{-\beta x^2}$$

   where the star denotes the convolution.