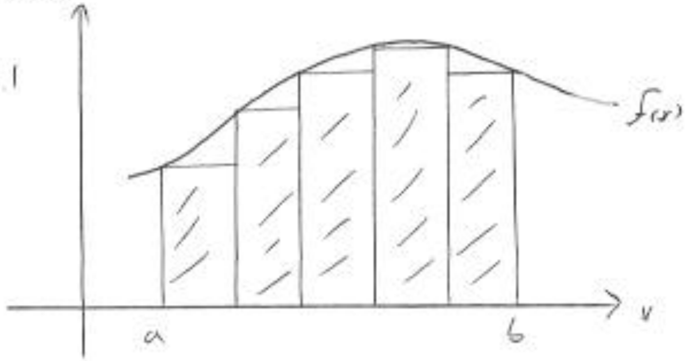
Pictures

I.1.1




$$\mathcal{R} = \sum_k f_k \, \Delta x_k$$

$\Delta x_k$ is width of $k^{th}$ interval

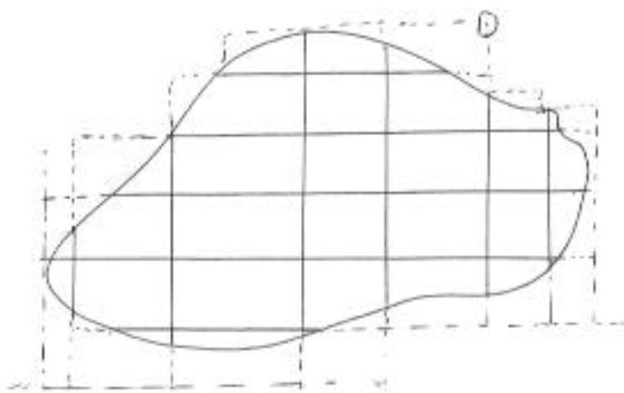
$f_k = \inf_{k^{th}\text{ interval}} f(x)$ i.e. the infimum of $f(x)$ in the $k^{th}$ interval.

$\int dx\, f(x) := \sup_{\text{all subdivisions}} \mathcal{R}$ i.e. the supremum of $\mathcal{R}$ over all subdivisions of $[a,b]$

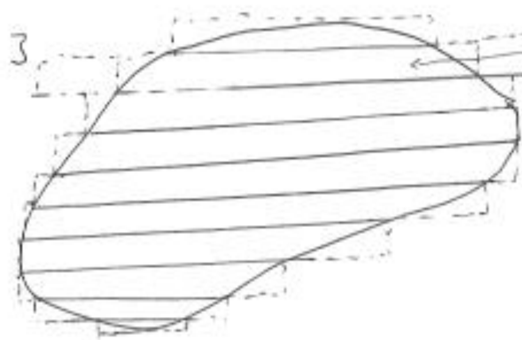
$:=$ is "defined as ~~by~~ being equal to"

---

I.1.2



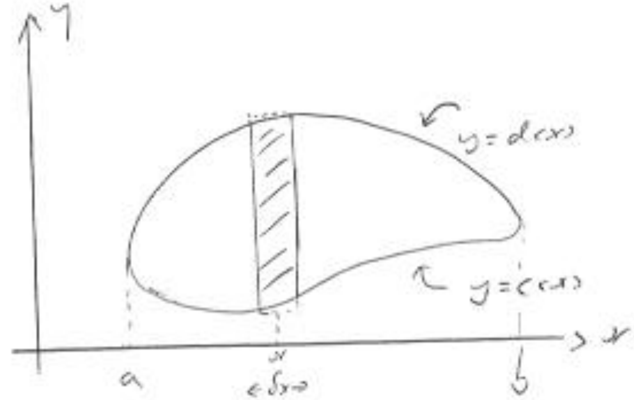

I.1.3



so sum along first to give series of slices.

PICTURES

I.1.4




sum associated with the slice at $x$ approaches

$$\delta x \int_{c(x)}^{d(x)} dy\, \phi(x,y)$$

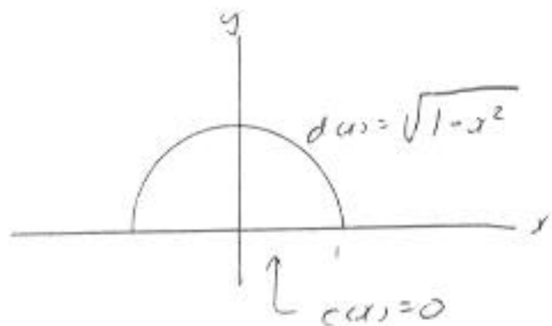
as $\delta x \to 0$.

so now the full integral is the sum of slices

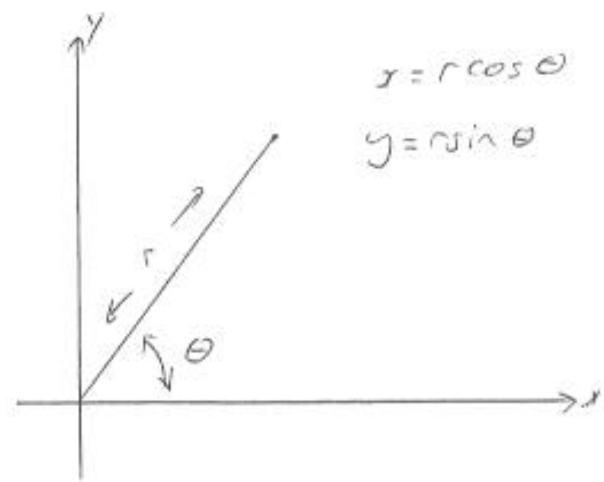
$$I = \int_a^b dx \int_{c(x)}^{d(x)} dy\, \phi(x,y)$$

& this is an iterated double integral.

I.1.5




I.1.6



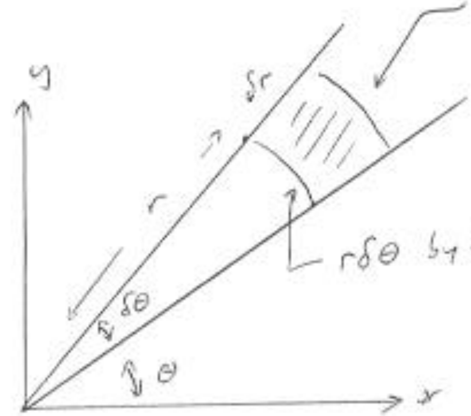

$x = r\cos\theta$

$y = r\sin\theta$

POLAR COÖRDINATES.

PICTURES

I.1.7




$(r+\delta r)\delta\theta = r\delta\theta + \delta r\delta\theta \approx r\delta\theta$ for $\delta r, \delta\theta$ small.

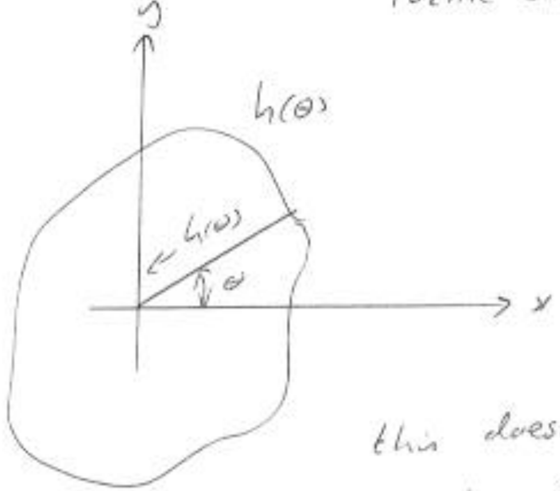
$r\delta\theta$ by def. of $\theta$ in radians

element of area for small $\delta r$, $\delta\theta$ is

$$\delta A = r\,\delta r\,\delta\theta$$

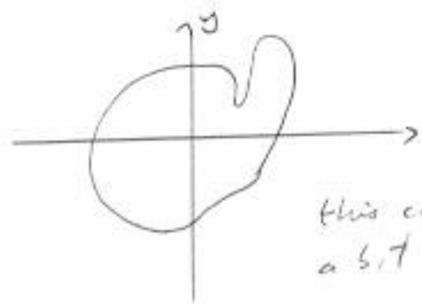
POLAR COÖRDINATES — element of area

I.1.8




$$\int_D dA\,\phi = \int_0^{2\pi} d\theta \int_0^{h(\theta)} dr\,\phi(r,\theta)\,r.$$

this doesn't work in so straightforward a way if the boundary can't be written as $r = h(\theta)$, for example, if it folds back

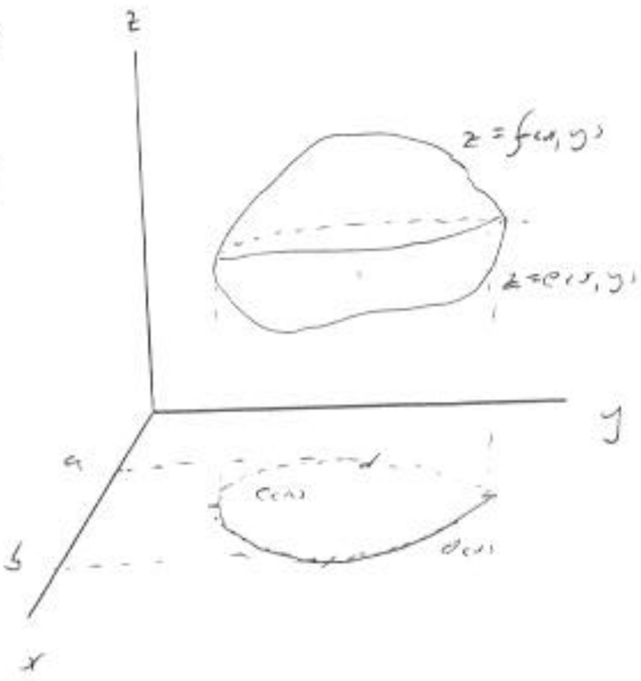



a similar caveat applies for $x, y$ coördinates.

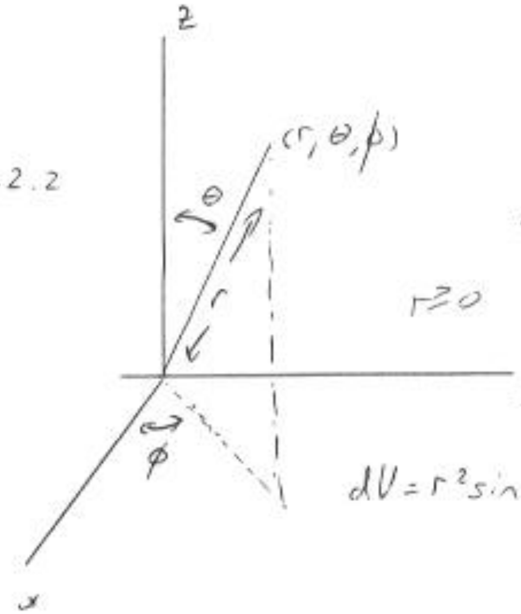
this case can be dealt with with a bit of care.

PICTURES

I.2.1




I.2.2




$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

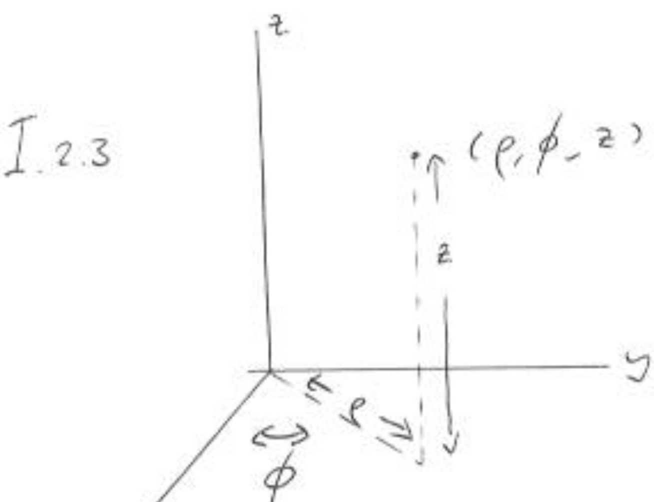
$r \geq 0 \quad 0 \leq \theta \leq \pi$ &

$\phi \equiv \phi + 2\pi$

$dV = r^2 \sin\theta \, dr \, d\theta \, d\phi$

SPHERICAL POLARS

I.2.3




$x = \rho\cos\phi$

$y = \rho\sin\phi$

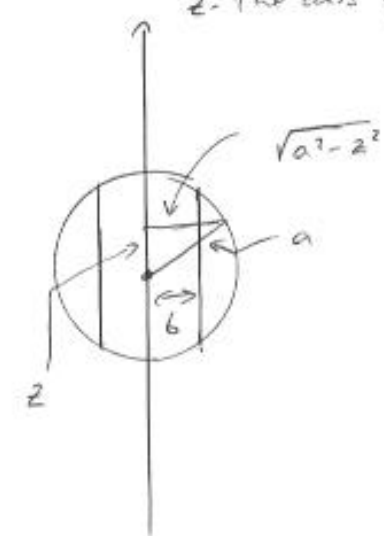
$z = z$

$dV = \rho \, d\rho \, d\phi \, dz$

CYLINDRICAL POLARS

I.2.4

z - the axis of the hole



PICTURES.

I.3.1

rtp

$$\underline{\nabla}.(\underline{F}\times\underline{G}) = (\underline{\nabla}\times\underline{F}).\underline{G} - \underline{F}.(\underline{\nabla}\times\underline{G})$$

$$= \partial_x(F_2G_3 - F_3G_2) + \partial_y(F_3G_1 - F_1G_3) + \partial_z(F_1G_2 - F_2G_1)$$

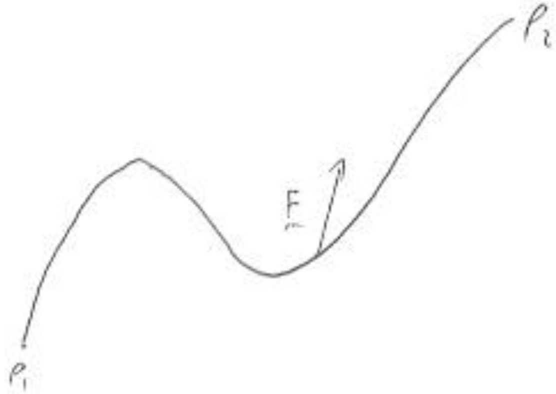
$$= G_1(\partial_y F_3 - \partial_z F_3) + F_1(\partial_z G_2 - \partial_y G_3) + G_2(\partial_z F_1 - \partial_x F_3) + F_2(\partial_x G_3 - \partial_z G_1) + G_3(\partial_x F_2 - \partial_y F_1) + F_3(\partial_y G_1 - \partial_x G_2)$$

$$= \underline{G}.(\underline{\nabla}\times\underline{F}) - \underline{F}.(\underline{\nabla}\times\underline{G})$$ as required.

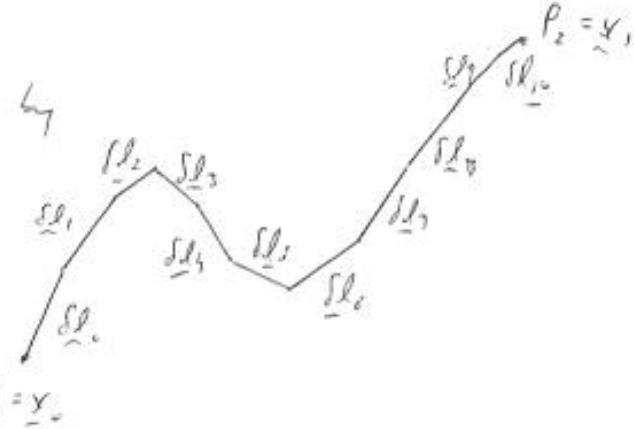
$$\underline{F}\times\underline{G} = \begin{vmatrix} i & j & k \\ F_1 & F_2 & F_3 \\ G_1 & G_2 & G_3 \end{vmatrix} = \begin{pmatrix} F_2G_3 - F_3G_2 \\ F_3G_1 - F_1G_3 \\ F_1G_2 - F_2G_1 \end{pmatrix}$$

PICTURES

I.4.1

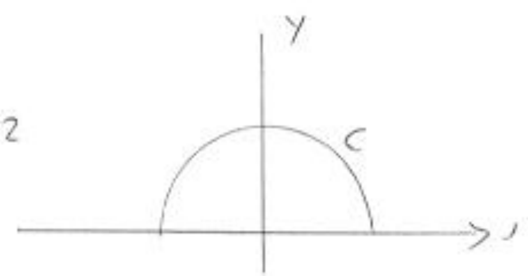



curve approximated by




& the line integral approximated by $\sum \underset{\sim}{F} \cdot \delta \underset{\sim}{l}$

I.4.2




we can actually do this integral directly

$$\int_C \underset{\sim}{F} \cdot d\underset{\sim}{l} = \int_C F_1 dx + \int_C F_2 dy$$

$$\int_C F_1 dx = \int_1^{-1} \tfrac{1}{2} y \, dx = -\tfrac{1}{2} \int_{-1}^{1} \sqrt{1-x^2} \, dx$$ $x = \sin\theta \quad dx = \cos\theta$, $\sqrt{1-x^2} = \cos\theta$

$$= \tfrac{1}{2} \int_{-\pi/2}^{\pi/2} \cos^2\theta \, d\theta = -\frac{\pi}{4}$$

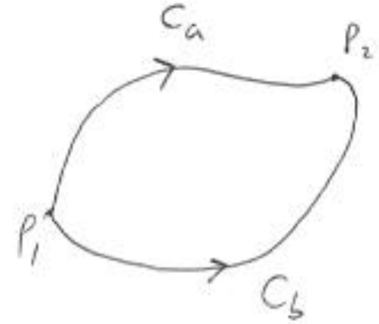
$$\int_C F_2 dy = -\int_0^1 \tfrac{1}{2} x \, dy - \int_1^0 dy \, \tfrac{1}{2} x$$ y increases & the decreases so $dy$ has different signs.

$$= -\int_0^1 \sqrt{1-y^2} \, dy = -\frac{\pi}{4}$$

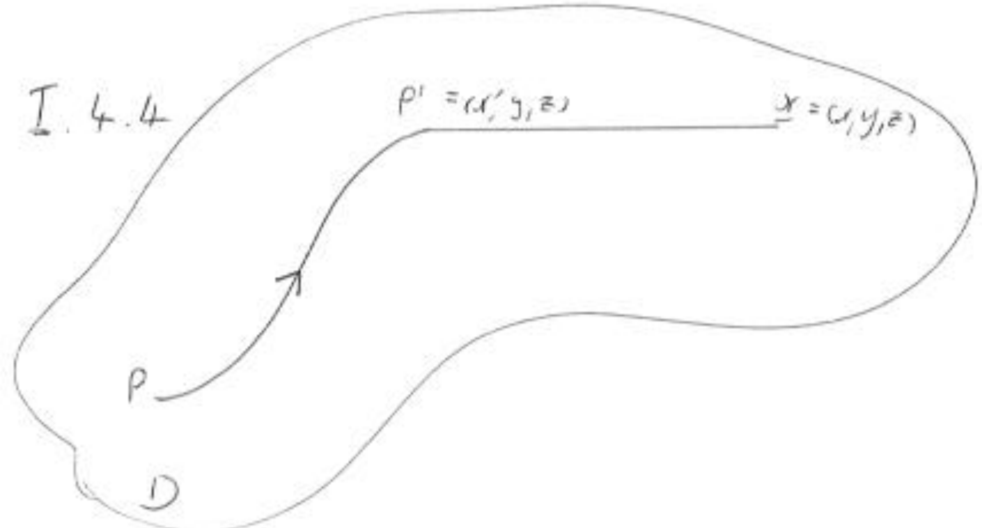
$\therefore \int_C \underset{\sim}{F} \cdot d\underset{\sim}{l} = -\frac{\pi}{2}$ the same answer but more straight forward using parameterization.

Pictures

I.4.3.




I.4.4




I.4.5.

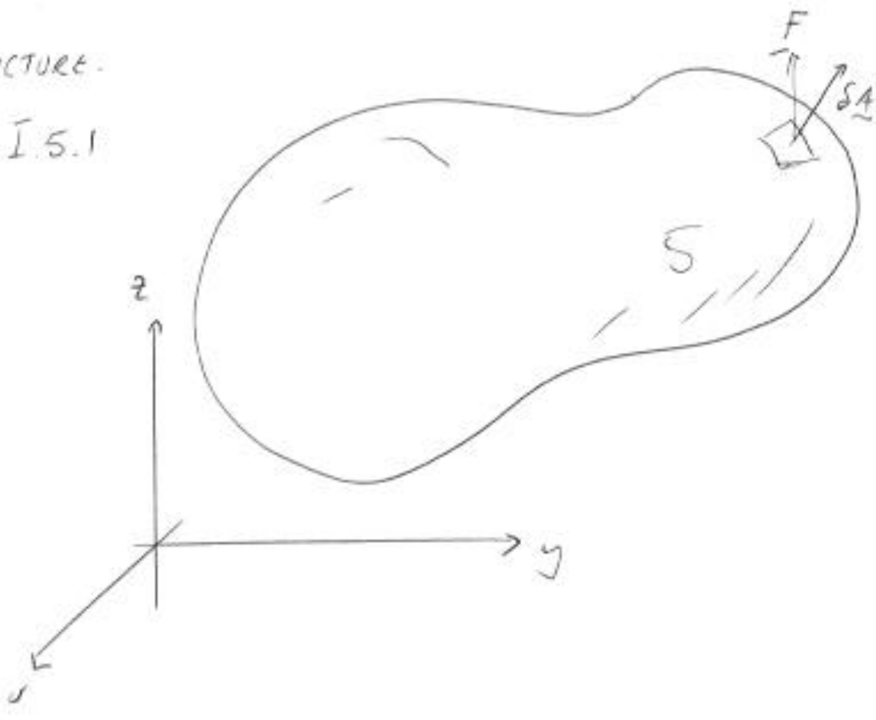
disconnected

connected but not simply connected

simply connected.

PICTURE.

I.5.1




I.5.2

oriented surface

this surface is orientable but in fact, considered as ~~an~~ oriented surfaces this is two oriented surfaces joined at the dotted line

I.5.2 cont.

mobius strip. - choose an orientation at one pt & try to extend is ~~up~~ around the strip; you will get a discontinuity

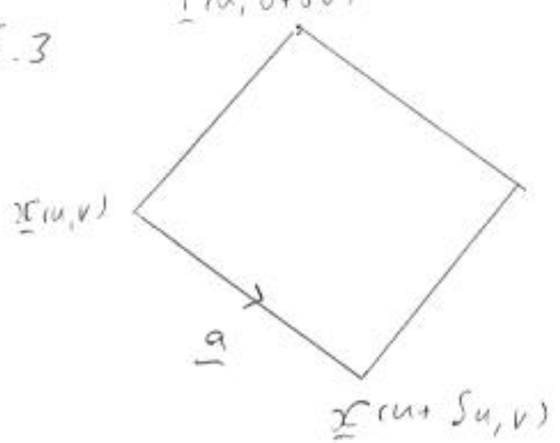
from above


ie. start here

& go around, it ends up with

PICTURES.

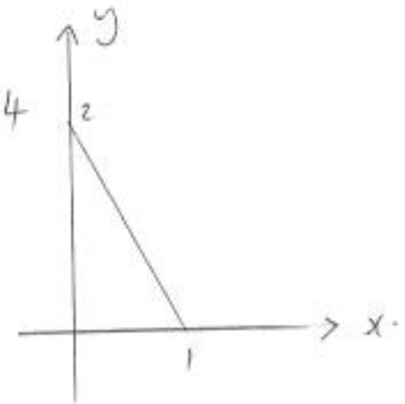
I.5.3



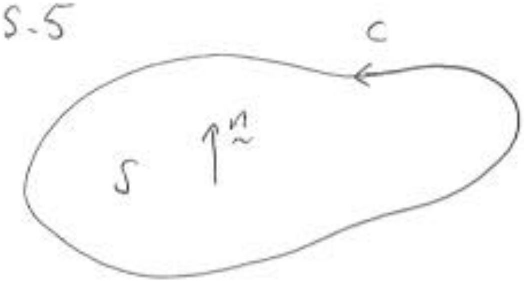

$$\underline{x}(u+\delta u, v) = \underline{x}(u,v) + \frac{\partial \underline{x}}{\partial u}\delta u + O(\delta u^2)$$
$$\underline{x}(u, v+\delta v) = \underline{x}(u,v) + \frac{\partial \underline{x}}{\partial v}\delta v + O(\delta v^2)$$

$$\Rightarrow \quad \underline{a} \approx \frac{\partial \underline{x}}{\partial u}\delta u$$
$$\underline{b} \approx \frac{\partial \underline{x}}{\partial v}\delta v.$$

I.5.4




I.5.5

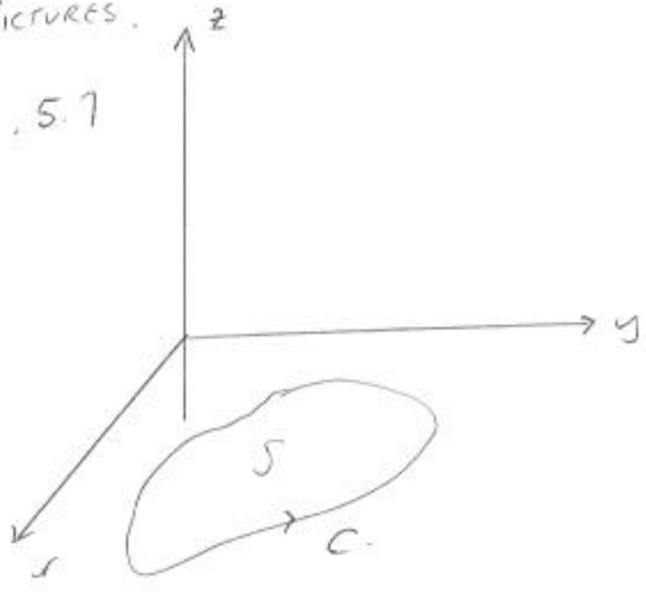



I.5.6

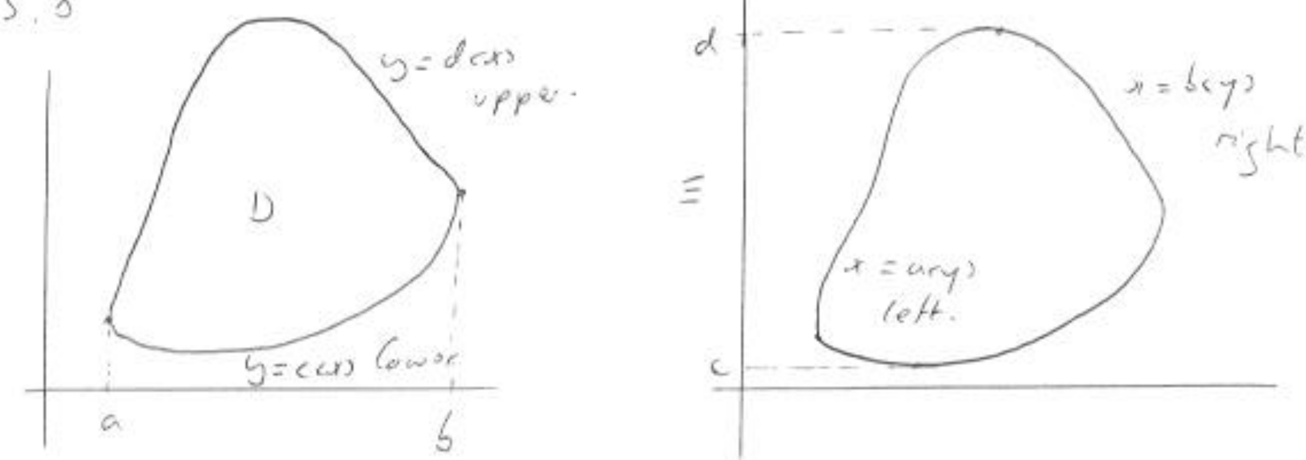

"if you walk along C in the direction given by the orientation & standing up as defined by the orientation of S, S is on your left."

PICTURES.

I.5.7




I.5.8

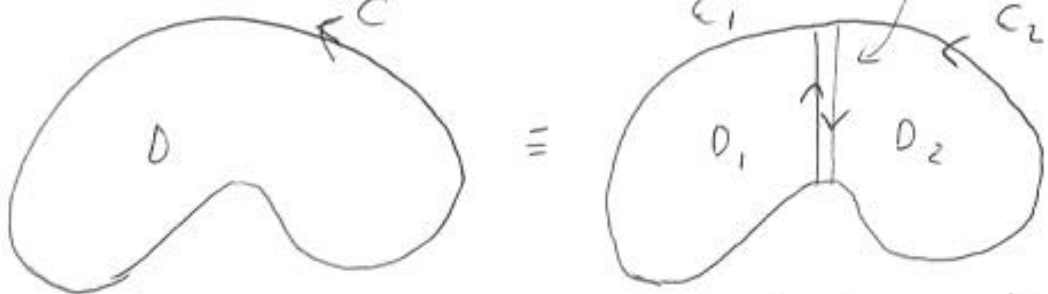



a SIMPLE REGION.

PICTURE.

I.5.9.

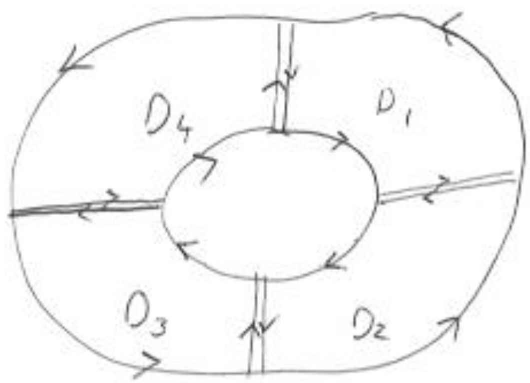
goes different directions along the join & hence cancels.




not simple

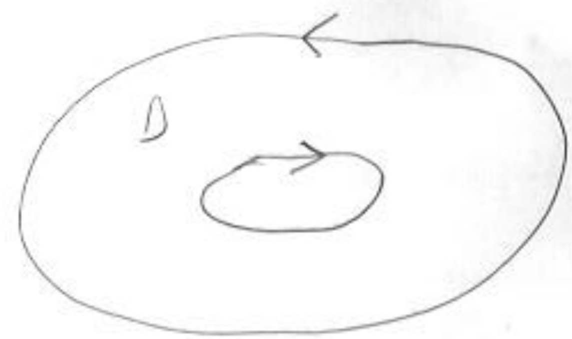
$D_1$ & $D_2$ simple

splitting up a region with an internal boundary into simple regions



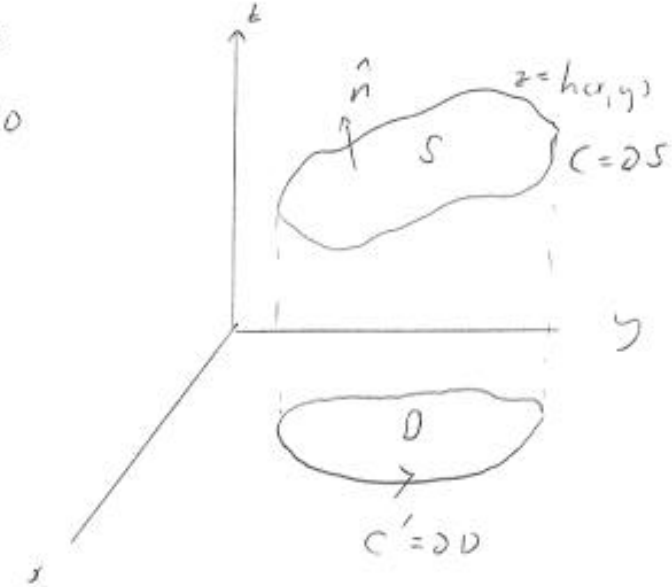

note the joins cancel & the internal boundary goes clockwise

≡



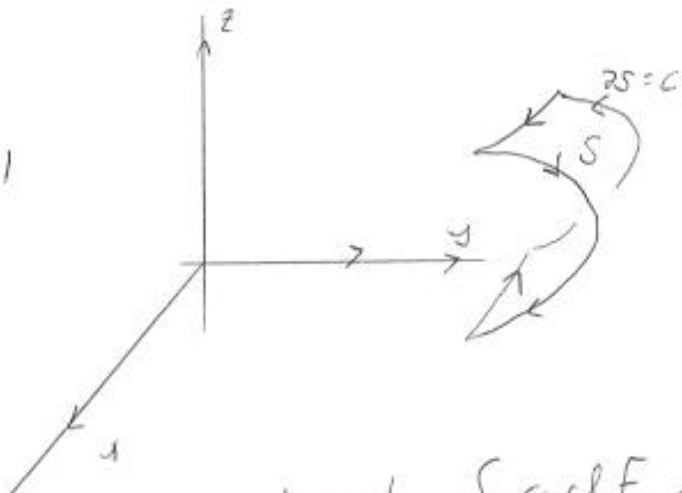
Pictures

I.5.10

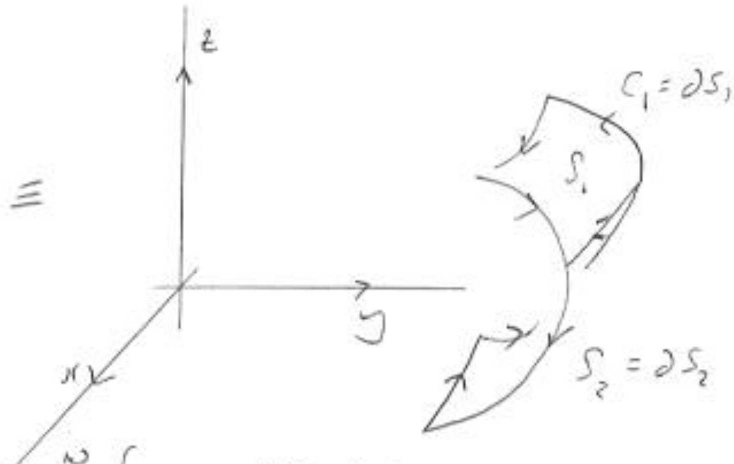



$S$ given by $\{(x,y,h(x,y)) \mid (x,y) \in D\}$

I.5.11

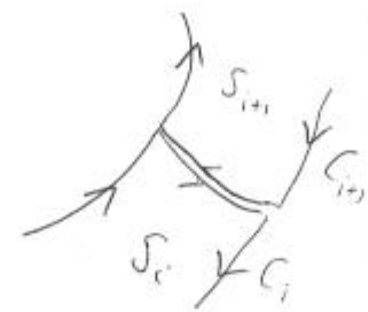



$\equiv$




obviously $\int_C \text{curl}\,\underline{F}\cdot d\underline{A} = \sum_{i=1}^{n} \int_{S_i} \text{curl}\,\underline{F}\cdot d\underline{A}$

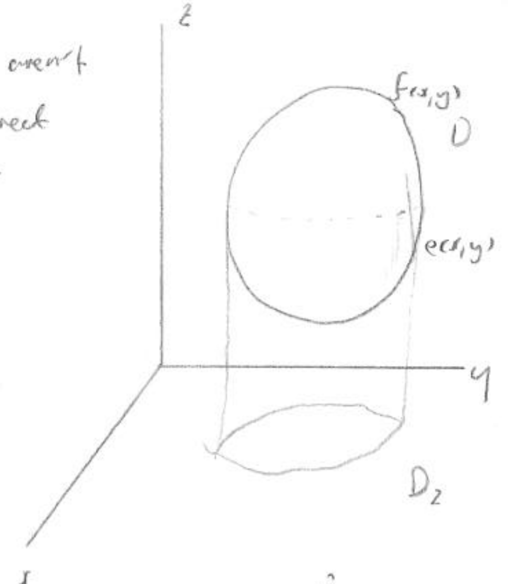
need to check $\int_C \underline{F}\cdot d\underline{l} = \sum_{i=1}^{n} \int_{C_i} \underline{F}\cdot d\underline{l}$ but, in fact like Green's theorem the joins cancel.



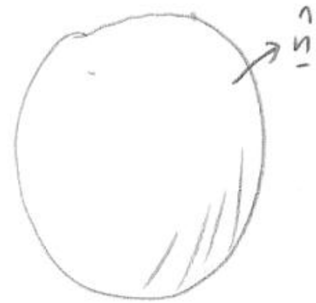
PICTURE

I.6.3.

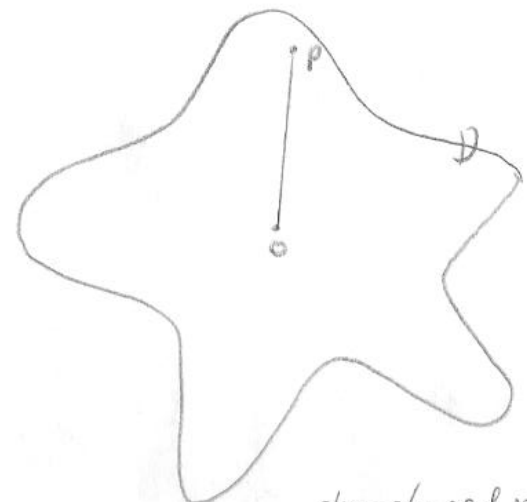
sorry the pictures aren't in the correct order.




I.6.1

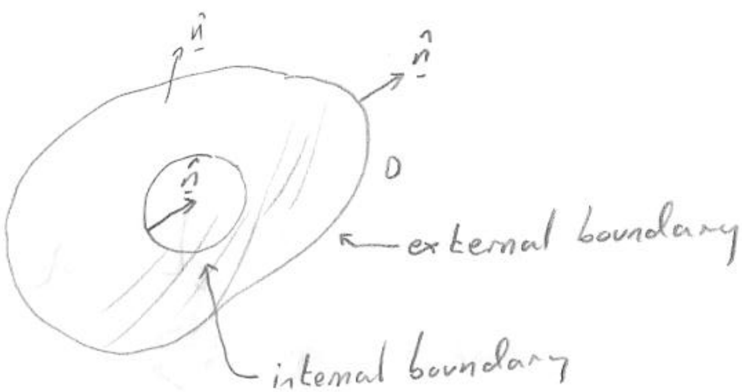



I.6.4

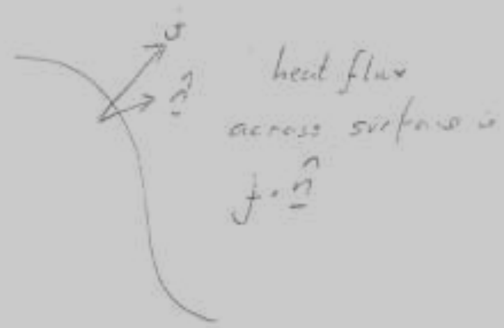



star-shaped region D
any pt. p can be joined to o by a straight line inside D.

I.6.2

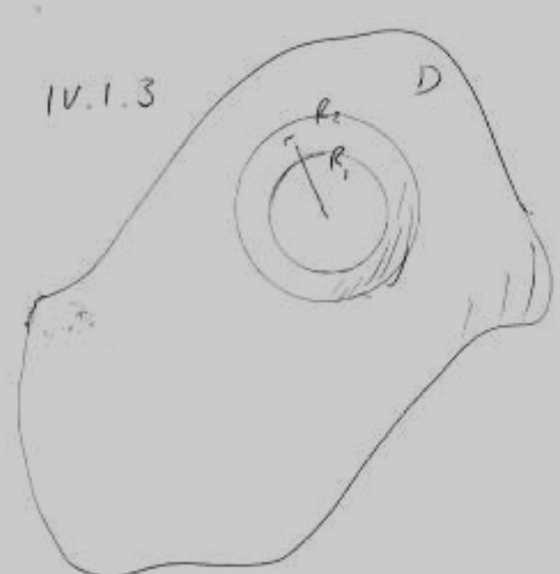


PICTURE IV.1.1

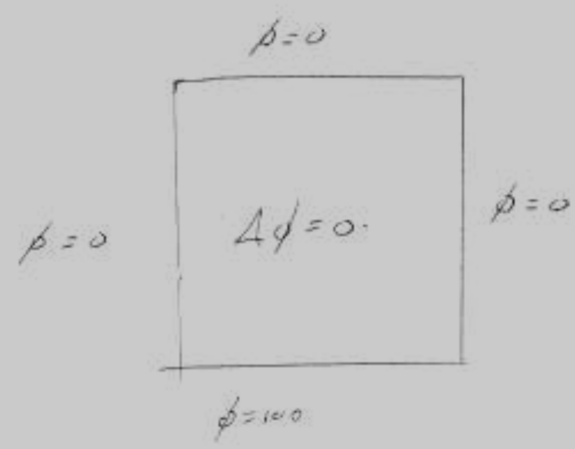



IV.1.2


IV.1.3



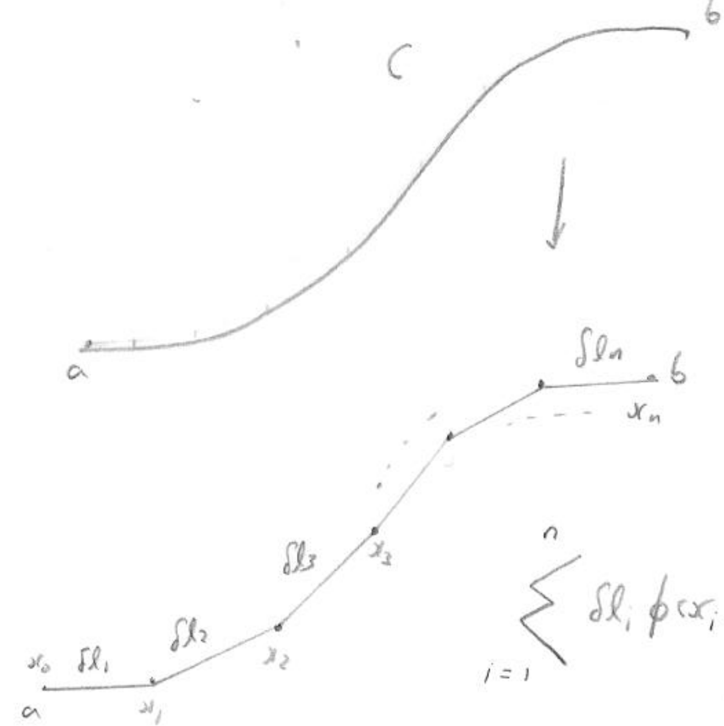

IV.1.4



PICTURE

I.6.5

$$\int_C dl\, \phi$$




$$\sum_{i=1}^{n} \delta l_i\, \phi(x_i) \longrightarrow \int dl\, \phi$$

& for $\int_S dS\, \phi$ the surface is approximated with infinitessimal elements $\delta S_i$

so $\sum \delta S_i\, \phi(x_i)$ where $\delta S_i$ is the area & $x_i$ is in $\delta S_i$.

$$\Downarrow$$

$$\int dS\, \phi$$