Fourier Analysis[1]

Distributions and the Fourier transform.

First, lets work out the **Fourier representation** of the delta function, that is, lets write it as a Fourier integral:

$$\delta(x) = \int_{-\infty}^{\infty} dk\widetilde{\delta(k)}e^{ikx} \tag{1}$$

where

$$\widetilde{\delta(k)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\delta(x)e^{-ikx} \tag{2}$$

but we can evaluate this integral easily because of the delta function, it gives $\widetilde{\delta(k)} = 1/2\pi$ and, hence,

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dke^{ikx} \tag{3}$$

This is an interesting and useful result, substituting $x - x'$ for x it gives

$$\delta(x - x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} dke^{ik(x-x')} \tag{4}$$

which is an orthonormality result analogeous to

$$\delta_{nm} = \frac{1}{2\pi}\int_{-\pi}^{\pi} dxe^{ix(n-m)} \tag{5}$$

for the periodic functions.

We can use this result to work out the Fourier integral of the constant function $f(x) = 1$:

$$\tilde{1} = \frac{1}{2\pi}\int_{-\infty}^{\infty} dke^{-ikx} = \delta(-k) = \delta(k) \tag{6}$$

so, together

$$\begin{aligned} \widetilde{\delta(k)} &= \frac{1}{2\pi} \\ \tilde{1} &= \delta(x) \end{aligned} \tag{7}$$

This is similar to what we saw for the Gauss function, the wider and flatter the function the narrower and pointier the transform and visa versa.

[1] Due to Conor Houghton

The Fourier integral formulas and the Plancherel formula

Now, we have used the Fourier integral formulas to write down the Fourier representation of the Dirac delta function, there are other ways to do this and they way we have done it, we certainly aren't in a position to use this formula to derive any part of the Fourier integral; however, it is interesting to see how the ideas hang together. Consider some function $f(x)$:

$$f(x) = \int_{-\infty}^{\infty} dx' f(x')\delta(x - x') \tag{8}$$

Now, substititute for the delta, this is a maneuver know as substituting a complete set:

$$f(x) = \int_{-\infty}^{\infty} dx' f(x') \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ik(x-x')} \tag{9}$$

and now move around the integrals a bit:

$$f(x) = \int_{-\infty}^{\infty} dk e^{ikx} \frac{1}{2\pi} \int_{-\infty}^{\infty} dx' f(x') e^{-ikx'} \tag{10}$$

which is precisely the Fourier integral formula.

Similarily, we can use this for the Plancherel formula:

$$\int_{\infty}^{\infty} dx |f(x)|^2 = \int_{\infty}^{\infty} dx f(x)^* f(x) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx' f(x)^* f(x') \delta(x - x') \tag{11}$$

and, again, substitute for the delta:

$$\begin{aligned}
\int_{\infty}^{\infty} dx |f(x)|^2 &= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx' \frac{1}{2\pi} \int_{-\infty}^{\infty} dk f(x)^* f(x') e^{ik(x-x')} \\
&= 2\pi \int_{-\infty}^{\infty} dk \frac{1}{2\pi} \int_{-\infty}^{\infty} dx f(x)^* e^{ikx} \frac{1}{2\pi} \int_{-\infty}^{\infty} dx' f(x') e^{-ikx'} \\
&= 2\pi \int_{-\infty}^{\infty} dk \widetilde{f(k)}^* \widetilde{f(k)} \qquad (12)
\end{aligned}$$