Fourier Analysis[1]

Distributions

The Heaviside function $\theta(x)$ is defined as

$$\theta(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases} \tag{1}$$

θ is related to the sign fxn

$$\epsilon(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases} \tag{2}$$

The question is, what is the integral

$$I = \int_{-\infty}^{\infty} dx \frac{d}{dx}\epsilon(x) \tag{3}$$

The first answer is to argue that the sign function is constant everywhere except at $x = 0$, therefore its derivative is zero except at one point, this is a set of measure zero and hence I is zero. Another arguement is to use the fundamental theorem of calculus:

$$I = \int_{-\infty}^{\infty} dx \frac{d}{dx}\epsilon(x) = \epsilon(\infty) - \epsilon(-\infty) = 1 - (-1) = 2 \tag{4}$$

We will develop a formalism, the theory of **distributions**, where the latter is the correct answer. We will do this in the traditional applied mathematics root, but, in fact, there is a beautiful mathematics theory of distributions which starts by trying to dualize the space of functions and, in this formulation, the fundamental theorem of calculus arguement above is very natural.

Consider the family of smooth functions

$$\epsilon_n(x) = \frac{2}{\pi}\tan^{-1} nx \tag{5}$$

These are sigmoid shaped function with

$$\int_{-\infty}^{\infty} dx \frac{d}{dx} e_n x = 2 \tag{6}$$

for all n and $\epsilon(x)$ can be viewed as the $n \to \infty$ limit of ϵ_n.

$$\text{Include a plot here} \tag{7}$$

Now, we define

$$\delta_n(x) = \frac{1}{2}\frac{d}{dx}\epsilon_n(x) = \frac{1}{\pi}\frac{n}{1 + n^2x^2} \tag{8}$$

[1] due to Conor Houghton

Then, we write

$$\delta(x) = \frac{1}{2}\frac{d}{dx}\epsilon(x) = \frac{d}{dx}\theta(x) \tag{9}$$

where $\delta(x)$ is the Dirac delta function. In this applied mathematics approach, it is the n goes to infinity limir of δ_n and is zero everywhere except $x = 0$ but has integral one from minus infinity to infinity.

In fact, the defining properties of the delta function are

$$\int_{-\infty}^{\infty} dx\delta(x)f(x) = f(0) \tag{10}$$

where $f(x)$ is any Schwartz function and $\delta(x) = 0$ for $x \neq 0$. Again, one approach to this, from an applied mathematics point of view, is to regard $\delta(x)$ as the ϵ goes to zero limit of

$$\delta_\epsilon(x) = \begin{cases} 1/\epsilon & 0 < x < \epsilon \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

Now, for sufficiently small ϵ $f(x) \approx f(0)$ for $0 < x < \epsilon$, so

$$\int_{-\infty}^{\infty} dx f(x)\delta_\epsilon(x) \approx f(0)\frac{1}{\epsilon}\int_0^\epsilon dx = f(0) \tag{12}$$

A practical approach to the delta function is to calculate its properties relative to its action under an integral. So,

$$\int_{-\infty}^{\infty} dx\delta(x-a)f(x) = f(a) \tag{13}$$

by a change of variable, $x' = x - a$ inside the integral. Similarily, using integration by parts and $\delta(x) = 0$ for $x \neq 0$

$$\int_{-\infty}^{\infty} \delta'(x)f(x) = \delta(x)f(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \delta(x)f'(x) = -f'(0) \tag{14}$$

and, using $y = ax$

$$\int_{-\infty}^{\infty} \delta(ax)f(x) = \frac{1}{a}\int_{-\infty}^{\infty} dy\delta(y)f(y/a) = \frac{1}{a}f(0) \tag{15}$$

where a is a positive constant, however, if a is negative then the change of variables changes the signs of the limits: if $a < 0$ $x = \infty$ implies $y = -\infty$ and so on. Hence

$$\int_{-\infty}^{\infty} \delta(ax)f(x) = -\frac{1}{a}f(0) \tag{16}$$

or, putting it all together

$$\int_{-\infty}^{\infty} \delta(ax)f(x) = \frac{1}{|a|}f(0) \tag{17}$$

Note that for $a = -1$, this formula gives $\delta(x) = \delta(-x)$, so the delta function is formally even.

Now, consider integral of the form

$$\int_{-\infty}^{\infty} dx\delta(h(x))f(x) \tag{18}$$

where $h(x)$ is some smooth function. If h has no zeros then integral is zero, so, for example

$$\int_{-\infty}^{\infty} dx\delta(1+x^2)f(x) = 0 \tag{19}$$

Assume that h has one zero at $x = x_1$, so $h(x_1) = 0$ and assume further that $h'(x_1) > 0$, this means that there is an interval (c, d) containing x_1 such that $h(x)$ is increasing on (c, d). Now, $\delta(x)$ is zero for $x \neq 0$ means that

$$\int_{-\infty}^{i} nfty\delta(h(x))f(x) = \int_{c}^{d} dx\delta(h(x))f(x) \tag{20}$$

and, since $h(x)$ is strictly increasing on (c, d) it is invertible on this interval, so we can do a change of variables to $y = h(x)$

$$\int_{c}^{d} dx\delta(h(x))f(x) = \int_{h(c)}^{h(d)} dy\frac{dx}{dy}\delta(y)f(x(y)) \tag{21}$$

so f is a function of y through $x = h^{-1}(y)$. Now, using the delta function to do the integral, and rewritting the Jacobian dx/dy=$1/h'$ we get

$$\int_{-\infty}^{\infty} \delta(h(x))f(x) = \frac{f(x_1)}{h'(x_1)} \tag{22}$$

Now, if $h'(x_1)$ had been negative everything would have been the same except there would have been an extra factor of minus one coming from changing around the limits, hence

$$\int_{-\infty}^{\infty} \delta(h(x))f(x) = \frac{f(x_1)}{|h'(x_1)|} \tag{23}$$

and finally, if there are n isolated zeros with non-zero derivative x_i where $i = 1, \ldots, n$

$$\int_{-\infty}^{\infty} \delta(h(x))f(x) = \sum_{i=1}^{n} \frac{f(x_i)}{|h'(x_i)|} \tag{24}$$

or, put another way

$$\delta(h(x)) = \sum_{i=1}^{n} \frac{\delta(x-x_i)}{|h'(x_i)|} \tag{25}$$

- **Example:**

$$\delta(x^2 - a^2) = \frac{1}{2|a|} \left[\delta(x-a) + \delta(x+a)\right] \tag{26}$$

so here $h(x) = x^2 - a^2$ with zeros at $\pm a$ and $h'(x) = 2x$

Finally, care needs to be taken with products, for example $\delta(x)^2$ and $\delta(x)\theta(x)$ are meaningless and $\delta(x-a)\delta(x-b)$ is zero for all $a \neq b$. The product rule do not work for discontinuities either, for example $\theta(x)^2 = \theta(x)$ so

$$\frac{d}{dx}\theta(x)^2 = \delta(x) \tag{27}$$

which is not what the product rule predicts. Similarly, for the sign function, $\epsilon(x)^2 = 1$ so

$$\frac{d}{dx}\epsilon(x)^2 = 0 \tag{28}$$

You might worry that $\epsilon(x)^2 = 1$ does not hold at $x = 0$, but no matter, this has no effect under an integral, so, as a distribution, $\epsilon(x)^2 = 1$.