PART I Fourier Analysis[1]

The Fourier integral

So far we have discussed the expansion of periodic functions; here we extend this discussion to functions which are not periodic. We will see that the expansion will go from a sum over a countably infinite set of basis function, to an integral over a continuum of basis function. It is possible to see why this will happen, the periodic function, with period l say, is expanded into basis functions which repeat after l, this does not require the basis functions to have period l, but, it does require that their period divides a whole number of times into l, reducing the set of basis functions to a countably infinite set.

The Fourier integral: definition

The expansion is an integral

$$f(x) = \int_{-\infty}^{\infty} dk \tilde{f(k)} e^{ikx} \tag{1}$$

where

$$\tilde{f(k)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk f(x) e^{-ikx} \tag{2}$$

Here, the basis functions are the so called **plane waves**, $\exp(ikx)$ and the Fourier coefficients are now functions, $\tilde{f(k)}$. The sum over a discrete index n has been replaced by an integral of a continuous index k. No proof is given here, we will revisit these formulas again after we have done the Dirac delta function, and, in the meantime, here is a formal arguement relating the Fourier integral to the Fourier transform. For a function $f(cx)$ with period l

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi inx/l} \tag{3}$$

where

$$c_n = \frac{1}{l} \int_{-l/2}^{l/2} dy f(y) e^{-2\pi iny/l} \tag{4}$$

or, putting this together

$$f(x) = \sum_{n=-infty}^{\infty} \frac{2\pi}{l} e^{2\pi inx/l} \frac{1}{\pi} \int_{-l/2}^{l/2} dy f(y) e^{-2\pi iny/l} \tag{5}$$

Now for l very large and $k = 2\pi/l$, the sum term above is a Riemann sum, giving

$$f(x) = \int_{-\infty}^{\infty} dk e^{ikx} \frac{1}{\pi} \int_{-l/2}^{l/2} dy f(y) e^{-iky} \tag{6}$$

[1] due to Conor Houghton

It should be noted that there are a number of different conventions for defining the Fourier integral; these vary in where the 2π goes. Some people share it between the $f(x)$ and $\tilde{f(k)}$ formula, giving factors of $1/\sqrt{2\pi}$ in front of both itegrals. Others send k to $2\pi k$ and $\tilde{f(k)}$ to $\tilde{f(k)}/2\pi$ so there are no constants multiplying either integral but the exponentials have 2πs in them.

The Fourier integral is not defined for all functions, the integrals defining $\tilde{f(k)}$ may be divergent. L^1 is often used, if

$$\int_{-\infty}^{\infty} dx|f(x)| < \infty \tag{7}$$

then the Fourier integral exists. However, the Fourier coefficient $\tilde{f(k)}$ may not itself have a Fourier integral: that is, the Fourier transform, the map from a function to its Fourier coefficient, is not closed on L^1. A space that works in this respect is the **Schwatz space** S, the space of rapidly decreasin functions, if $f \in S$ then $\tilde{f} \in S$. f is **rapidly decreasing** if it is infinitely differentiable and

$$\sup \left| x^r \frac{d^s f}{dx^2} \right| < \infty \tag{8}$$

for all positive r and s and with the sup taken over all x.

- **Example:** Consider the square wave packet

$$f(x) = \begin{cases} 1 & -1 < x < 1 \\ 0 & |x| > 1 \end{cases} \tag{9}$$

This is an example of an L^1 function which is not in S. Now,

$$\tilde{f(k)} = \frac{1}{2\pi}\int_{-1}^{1} dx e^{-ikx} = \left.\frac{e^{-ikx}}{-2\pi ik}\right|_{-1}^{1} = \frac{\sin k}{\pi k} \tag{10}$$

and so

$$f(x) = \int_{-\infty}^{\infty} dk \frac{\sin k}{\pi k} e^{ikx} \tag{11}$$

Hence, for example, putting $x = 0$ gives

$$\pi = \int_{-\infty}^{\infty} dk \frac{\sin k}{k} \tag{12}$$

This is Dirichlet's integral and can also be calculated using contour integration. At$x = \pm 1$, f has a jump discontinuity, the Fourier integral averages at a jump, just like the Fourier series, we can see this in this case, at $x = 1$:

$$\int_{-\infty}^{\infty} dk \frac{\sin k}{\pi k} e^{ik} = \int_{-\infty}^{\infty} dk \frac{\sin k}{\pi k}(\cos k + i \sin k) \tag{13}$$

Now dropping the odd integral over $sin^2 k/k$ we get

$$\int_{-\infty}^{\infty} dk \frac{\sin k}{\pi k} e^{ik} = \int_{-\infty}^{\infty} dk \frac{\sin k \cos k}{\pi k} = \int_{-\infty}^{\infty} dk \frac{\sin 2k}{2\pi k} = \frac{1}{2} \tag{14}$$

since it is in the Dirichlet integral form.

Parseval's Theorem

If

$$\int_{-\infty}^{\infty} dx|f(x)|^2 dx < \infty \tag{15}$$

then

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dx|f(x)|^2 = \int_{-\infty}^{\infty} dk|\tilde{f(k)}|^2 \tag{16}$$

If $f \in S$ this result is sometimes called the Plancherel formula. We will prove this formula later, at the moment we have no formula analogous to

$$\int_{-\pi}^{\pi} dx e^{i(n-m)x} = 2\pi\delta_{nm} \tag{17}$$

We will see that the Dirac delta function, which we look at next, provides such a formula.

- **Example:** Applying this formula for the square wave packet above, we get

$$\int_{-\infty}^{\infty} dk \frac{\sin^2 k}{k^2} = \pi \tag{18}$$