Introduction to number theory
Exercise sheet 5

https://www.maths.tcd.ie/~mascotn/teaching/2020/MAU22301/index.html

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Answers are due for Wednesday December 2nd, 2PM.
The use of electronic calculators and computer algebra software is allowed.

Exercise 1 *Reduction (15 pts)*

Find a reduced quadratic form equivalent to the form $22x^2 - 16xy + 3y^2$.

Exercise 2 *Primes of the form... (85 pts)*

1. (5 pts) Let $p \neq 2, 11$ be prime. Prove that $\left(\frac{-11}{p}\right) = \left(\frac{p}{11}\right)$.

2. (50 pts) Let $p \in \mathbb{N}$ be an odd prime. Prove that p is of the form $x^2 + xy + 3y^2$ (with $x, y \in \mathbb{Z}$) if and only if $p = 11$ or $p \equiv 1, 3, 4, 5 \text{ or } 9 \pmod{11}$.

   *You are **not** allowed to use the theorem giving the list of D such that $h(D) = 1$.*

3. (30 pts) Let $p \in \mathbb{N}$ be an odd prime. Using only a minimum amount of computations, prove that p is of the form $15x^2 - 17xy + 5y^2$ (with $x, y \in \mathbb{Z}$) if and only if $p = 11$ or $p \equiv 1, 3, 4, 5 \text{ or } 9 \pmod{11}$.

   If you use the results of the previous question, there is a way to solve this question with almost no computations; this is what you must do in order to get all the marks for this question.

These were the only mandatory exercises, that you must submit before the deadline. The following exercises are not mandatory; they are not worth any points, and you do not have to submit them. However, I highly recommend that you try to solve them for practice, and you are welcome to email me if you have questions about them. The solutions will be made available with the solution to the mandatory exercises.

Exercise 3 *Class numbers*

Compute the class number $h(D)$ for

1. $D = -116$,
2. $D = -47$.

Exercise 4 *Easy cases of the class number 1 problem*

1. Let $n \in \mathbb{N}$ be congruent to 1 or 2 mod 4. Prove that $h(-4n) = 1$ if and only if $n < 3$.

   Hint: Imagine that you apply the method seen in class to compute $h(-4n)$. What happens when $n \geqslant 3$?

2. Let $n \in \mathbb{N}$. Prove that if $h(-4n + 1) = 1$, then $n = 2$ or n is odd.